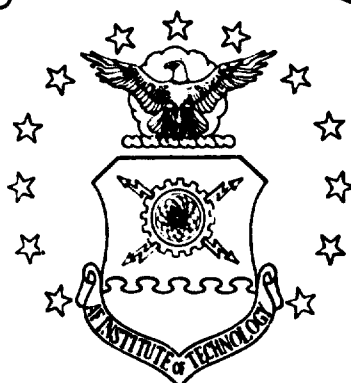# AIR FORCE INSTITUTE OF TECHNOLOGY

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AIR UNIVERSITY
UNITED STATES AIR FORCE

APPLICATION OF DIFFERENTIAL GAME THEORY
TO PURSUIT-EVASION PROBLEMS OF
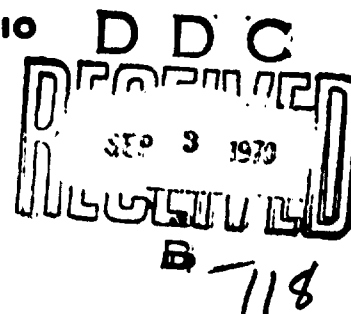TWO AIRCRAFT

DISSERTATION

DS/MC/67-1   William L. Othling, Jr.
              Major   USAF

SCHOOL OF ENGINEERING

WRIGHT-PATTERSON AIR FORCE BASE, OHIO

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APPLICATION OF DIFFERENTIAL GAME THEORY

TO PURSUIT-EVASION PROBLEMS OF

TWO AIRCRAFT

DISSERTATION

Presented to the Faculty of the School of Engineering of

the Air Force Institute of Technology

Air University

in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

in Aerospace Engineering

By

William L. Othling, Jr., M.S.

Major   USAF

June 1970

PREFACE

This work is the result of my attempt to approach the two aircraft combat simulation problem from a game theoretic point of view. The acceptance and support of this approach by the Air Force Flight Dynamics Laboratory is gratefully acknowledged. I am indebted to Professors Roger W. Johnson and Gerald M. Anderson at the Air Force Institute of Technology for their advice and encouragement. Our association has been a rewarding personal experience for me.

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ABSTRACT

The pursuit-evasion aspect of the two aircraft combat problem is introduced as a fixed time, zero sum, perfect information differential game. The purpose of this dissertation is to solve this differential game problem and to obtain closed-loop guidance or control laws. A realistic aircraft model is presented for which a solution of this combat problem is desired. Because of the non-linear dynamics associated with this model, an optimal closed-loop solution cannot be obtained. Three additional simplified aircraft models are introduced as approximations to the realistic model. Optimal solutions and closed-loop control laws are obtained for each of these models.

Analysis of the solutions and control laws reveals that they are characteristically similar and to a certain extent independent of the aircraft model. This enables the formulation of an approximate closed-loop control law for use with the original realistic model. The optimality of this control law is established by applying it to the pursuer in a differential game problem while the evader determines the best open-loop evasive strategy.

I. INTRODUCTION

. In recent years, the problem of determining the performance requirements for a superiority fighter aircraft has received the attention of many researchers. The basic difficulty is the lack of numerical measures to evaluate the effectiveness of one aircraft when pitted against another in a competitive or combative situation. During an aircraft combat engagement there are periods when one or both aircraft may be passive, aggressive or evasive depending upon the relative capabilities and positions of the aircraft and the desires of the pilots.

This dissertation considers the pursuit-evasion aspect of the combat problem. One approach to the solution of this problem is to determine the control of one aircraft which pursues in some optimal manner another aircraft which either employs a predetermined control law or follows a pre-specified trajectory. The difficulty associated with this approach is that the optimal solution to one problem is not optimal if either the pre-specified control law or trajectory is changed. Simply stated this means that no one guidance scheme is optimal against all types of evasion. If optimal pursuit and optimal evasion can be considered together, it becomes possible to derive a numerical measure reflecting the capabilities of the two aircraft. The problem of determining the optimal controls for such a problem is a differential game problem. In order to have any practical application, the solution to this problem must provide feedback strategies, or what is equivalent, a continuous real-time solution of open-loop strategies.

It is the intent of the dissertation to determine optimal
or near optimal feedback strategies for a pursuit-evasion differ-
ential game between two aircraft. The solubility of this problem
is dependent on the model chosen to represent the aircraft. An
important aspect of this study is the investigation of different
aircraft models, characterized by various assumptions and approxi-
mations, in an effort to determine optimal or "near" optimal
closed-loop feedback strategies.

The purpose of this study and the realistic aircraft model to
be considered is discussed in Chapter II. In Chapter III, a
differential game problem is defined. Necessary and sufficient
conditions for a solution of this problem are presented. Chapters
V thru VII are concerned with the solutions to the pursuit-evasion
differential game problems characterized by different aircraft models.
The solutions and control laws derived in these chapters are applied
to two realistic problems in Chapter VIII. A closed-loop control
law for application with the realistic aircraft model is synthesized
and discussed in Chapter IX. Conclusions and recommendations are
presented in Chapter X.

It is believed that there are two main contributions resulting
from this research. For the first time, a realistic aircraft
pursuit-evasion problem is presented and solved. The solution to
this problem has important application in the areas of aircraft
design and performance, and aircraft tactics. The second contribution
is the demonstration that reasonable solutions to differential game
problems can often be obtained through analysis of simplified models.
It is believed that this differential game modeling approach can be

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applied to a much larger class of differential game problems involving realistic aircraft models.

II. STATEMENT OF THE PROBLEM

Purpose of Dissertation and Approach

The purpose of this dissertation is to solve the two-aircraft pursuit-evasion differential game and to obtain optimal closed-loop control laws or strategies for the two aircraft. The extent to which optimal closed-loop strategies can be determined is highly dependent on the dynamics chosen to represent the aircraft. In the event optimal feedback solutions cannot be found, "near" optimal or approximately optimal feedback strategies are sought.

In approaching this task, a standard aircraft model is defined. Application of the necessary conditions for a saddle point solution yields a two point boundary value problem, for which closed-loop control laws could not be found. Open-loop controls and solutions are obtained, however, by means of iterative numerical techniques. The next step is to determine what simplifications can be made to the standard model dynamics so that closed-loop strategies can be determined. Three simplified aircraft models are considered and closed-loop strategies are obtained for each.

It is discovered that the optimal paths for the differential game problem solutions with the simplified models exhibit similar characteristics to the optimal paths obtained for the standard models. The similarity of solutions suggests a method of synthesizing a feedback control law for the standard model. The optimality of this synthesized control law is evaluated by formulating a one-sided differential game or optimal control problem in which the path of the pursuer is determined through application of this control law

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while the path of the evader is optimized against it.

Differential Game Problem

The pursuit-evasion problem is to be formulated as a two player, zero sum, perfect information differential game. The game is assumed to take place in the vertical plane and terminates when the terminal manifold defined by

$$\psi \left[t_f, \underline{x} \left(t_f \right) \right] = T - t_f = 0$$

is reached. T is a fixed specified time, t_f is the time when the terminal manifold is reached and $\underline{x} \left(t_f \right)$ is an n dimensional vector representing the state of the game evaluated at the terminal manifold. The range (R) between the two vehicles at termination is the payoff and is denoted as J. It becomes obvious, then, that in this pursuit-evasion game the evader will strive to maximize the range at termination and the pursuer will want to minimize R at termination. Thus, one player's gain is the other player's loss. Such games are referred to as zero sum differential games. The game is assumed to be a game of perfect information, where each player (aircraft) knows the present state of the game $\underline{x} \left(t \right)$ along with the dynamics and capabilities of his opponent.

Aircraft Dynamics

The following assumptions are made in regard to the aircraft dynamics:

      (a) The aircraft are considered to be point masses.

      (b) The earth is flat and the acceleration of gravity is constant.

(c) The thrust is constant and tangent to the flight path.

(d) The aircraft weight is constant.

In light of these assumptions the aircraft equations of motion

$$\dot{x} = V \cos \gamma$$

$$\dot{h} = V \sin \gamma$$

$$\dot{V}_x = (G/W) \ [(T - D) \cos \gamma - L \sin \gamma]$$

$$\dot{V}_h = (G/W) \ [(T - D) \sin \gamma + L \cos \gamma] - G$$

where γ is defined by

$$\gamma = \tan^{-1} (V_h/V_x)$$

denotes the horizontal distance, h the altitude, V_x and V_h are the

h components of velocity V, γ the flight path inclination, W

weight, D the drag, L the lift, T the thrust and G the acceleration

gravity. The forces acting on the aircraft are shown in Figure 1.
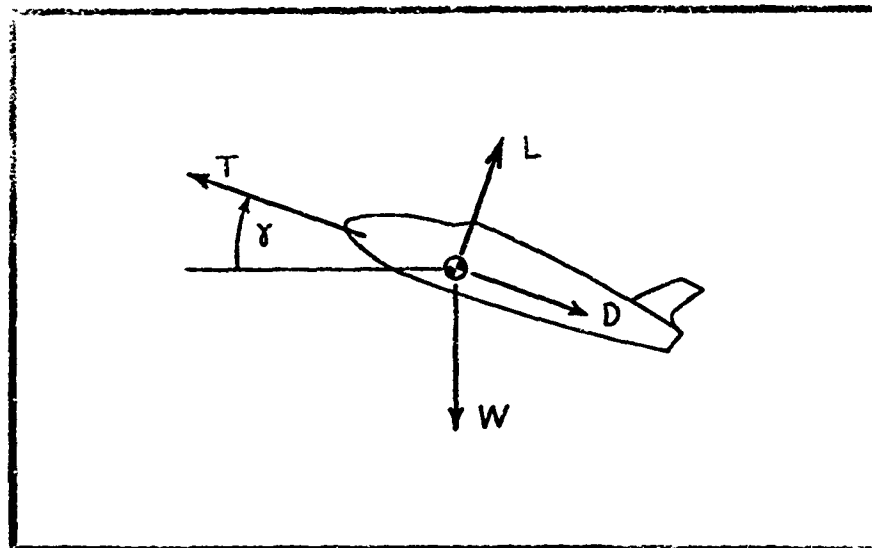
Figure 1. Aircraft Forces

The aerodynamic forces are defined by

$$D = 1/2\rho \; V^2 \; S \; C_D \qquad , \qquad L = 1/2\rho \; V^2 \; S \; C_L$$

where ρ is the air density, S a reference area, C_D the drag coefficient and C_L the lift coefficient.

In general, (Ref [11]), C_D and C_L are functions of the angle of attach α, the Mach number M and the Reynolds number Re with the functional relationships

$$C_D = C_D \; (\alpha, \; M, \; Re) \qquad , \qquad C_L = C_L \; (\alpha, \; M, \; Re)$$

For angles of attack below the stalling point α can be eliminated from the above relationships to yield

$$C_D = C_D \; (C_L, \; M, \; Re)$$

which is called the drag polar. For relatively constant values of M and Re the dependence of C_D on M and Re can be neglected. If the assumption is made that the drag polar is parabolic with constant coefficients, the drag and lift coefficients satisfy the relationship

$$C_D = C_{DO} + k \; C_L^2$$

where C_{DO} is the zero-lift drag coefficient and k the induced drag factor. The assumption that the drag polar is parabolic is in many cases a good approximation to the experimental polar. Its accuracy depends on the lift coefficient as well as the aircraft configuration. This approximation is extremely useful in analytical work and is therefore widely used.

The additional assumption of constant dynamic pressure ($Q=1/2\rho \; V^2$) is made. This assumption is not severe for the pursuit-evasion game where velocity and altitude changes are minimal. This leads to the

tion of motion which are referred to in this study as the standard
aircraft model.

$$\dot{x} = V_x = V \cos \gamma$$

$$\dot{h} = V_h = V \sin \gamma$$

$$\dot{V}_x = (Q\ S/m)[(C_T - k\ C_L^2) \cos \gamma - C_L \sin \gamma]$$

$$\dot{V}_h = (Q\ S/m)[(C_T - k\ C_L^2) \sin \gamma + C_L \cos \gamma] - G$$

where C_T is a coefficient formed from $Q\ S\ C_T = T - Q\ S\ C_{DO}$.

Acceleration Vectograms

In (Ref [10]) Isaacs introduces the concept of a vectogram to
describe as a function of the state of the game the allowable choice
controls for each player. For example, simple planar motion with
the velocity direction as the control can be depicted by the circular
diagram shown in Figure 2.
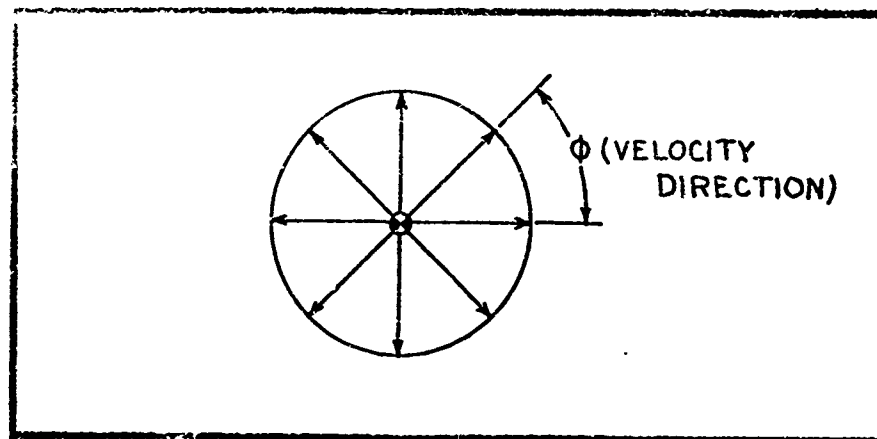
Figure 2. Circular Vectogram

The concept of control vectograms has been a convenient one

during this study because it provides a visual description of the force relationships of the problem. For this study, the definition of a vectogram is modified to include both the thrust and aerodynamic forces acting on the aircraft. Gravitational forces are not included. The acceleration vectogram for the standard aircraft model is shown in Figure 3.

From a vectogram point of view, the differential game problem can be viewed as the problem of determining how to employ the available forces in order to best pursue or evade the opposing player. It is obvious that a continuous compromise or tradeoff between force
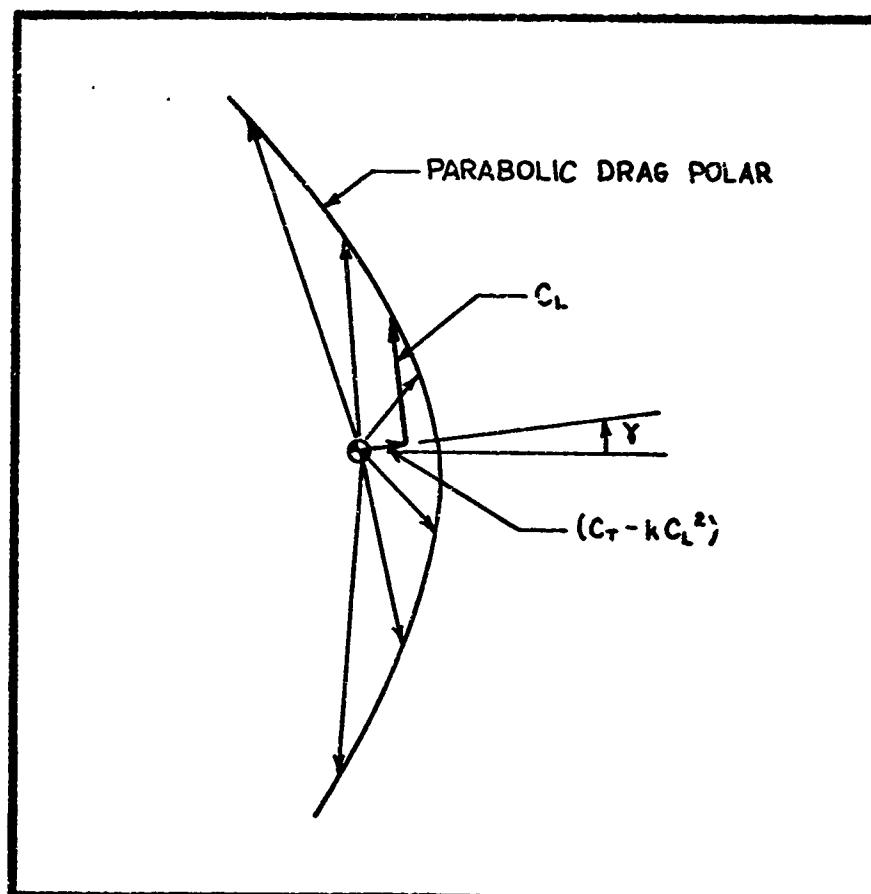
Figure 3. Acceleration Vectogram for Standard Aircraft Model

magnitude and direction must be made. The problem of determining how
to employ the available forces is difficult because the force or
acceleration magnitude available in a given direction varies with the
aircraft's flight path direction. The approach taken in this study
is to consider various simplifications of the standard aircraft model
acceleration vectogram that will yield a problem for which closed-loop
solutions can be found.

III. THEORY OF DIFFERENTIAL GAMES

The pursuit-evasion problems treated in this study are formulated as fixed time, zero sum, perfect information differential games. The purpose of this chapter is to define mathematically this class of differential games and to present necessary and sufficient conditions for solutions to these problems. The basis of this chapter can be found in References [1], [2], and [4].

Mathematical Formulation

The differential game problem treated in this dissertation is defined by the dynamic system

$$\underline{\dot{x}} = \underline{f}\,(\underline{x},\, u,\, v,\, t) \qquad ; \qquad \underline{x}\,(t_o) = \underline{x}_o \qquad (1)$$

where $\underline{x}$ is an n-vector, u is the scalar pursuer control and v is the scalar evader control. The goal is to find the controls u^* and v^* such that

$$\psi\,(\underline{x}\,(t_f),\, t_f) = 0$$

and the performance criteria

$$J = \phi\,(\underline{x}\,(t_f),\, t_f) + \int_{t_o}^{t_f} L\,(\underline{x},\, u,\, v,\, t)\; dt \qquad (2)$$

satisfies

$$J\,(u^*,\, v) \le J\,(u^*,\, v^*) \le J\,(u,\, v^*) \qquad (3)$$

If u^* and v^* can be found, the pair $(u^*,\, v^*)$ is called a saddle point of the game and $J\,(u^*,\, v^*)$ is called the value of the game. Eq (3) is equivalent to the following equations

$$J\ (u^*,\ v^*) = \underset{u \varepsilon U}{\text{Min}}\ J\ (u,\ v^*)$$

$$\approx \underset{v \varepsilon V}{\text{Max}}\ J\ (u^*\ v) \tag{4}$$

$$= \underset{u \varepsilon U\ v \varepsilon V}{\text{Min Max}}\ J\ (u,\ v)$$

where U and V are admissible sets for u and v.

Eq (4) and hence the existence of a saddle point are dependent on the condition that

$$\underset{u \varepsilon U\ v \varepsilon V}{\text{Min Max}}\ J\ (u,\ v) = \underset{v \varepsilon V\ u \varepsilon U}{\text{Max Min}}\ J\ (u,\ v) \tag{5}$$

Necessary Conditions

A necessary condition for a saddle point solution of the differential game problem defined by Eqs (1), (2) and (3) is that the Hamiltonian defined by

$$H\ (t,\ \underline{x},\ \underline{\lambda},\ u,\ v) = \underline{\lambda}^T\ \underline{f} + L \tag{6}$$

must be minimized over the set of admissible u and maximized over the set of admissible v and that

$$H^* = \underset{v\ \ u}{\text{Max Min}}\ H = \underset{u\ \ v}{\text{Min Max}}\ H \tag{7}$$

$\underline{\lambda}$ is the n-dimensional costate vector and the costate differential equations are

$$\underline{\dot{\lambda}}^T = -H_{\underline{x}} \tag{8}$$

The transversality conditions are given by

$$\underline{\lambda}^T\ (t_f) = \phi_{\underline{x}}\ (t_f) \tag{9}$$

Eq (7) implies that the maximization and minimization processes commute, which is not generally true. It is true, however, if H

can bo separated into two functions, one of which is indopendent of u and the other independent of v. For example

$$H(t, \underline{x}, \underline{\lambda}, u, v) = H^1(t, \underline{x}, \underline{\lambda}, u) + H^2(t, \underline{x}, \underline{\lambda}, v)$$

For the problems considered in this dissertation $\underline{f}$, L, and hence H are separable. This insures that the minimizing u and maximizing v provide a saddle point of H at each point on the optimal path. Unfortunately, separability of H does not imply separability of J and, therefore, solutions to the two point boundary value problem given by Eqs (6), (8), and (9), may not necessarily satisfy the saddle point conditions given by Eq (3). A procedure for verifying the saddle point conditions are given in the section on sufficient conditions.

Necessary conditions require that the Hamiltonian be minimized over the set of admissible u and maximized over the set of admissible v. If H is linear in the control variables and if $\partial H/\partial u$ and $\partial H/\partial v$ are equal to zero the Hamiltonian is independent of u and v and it is not possible to maximize or minimize H with respect to these controls. Extremal arcs on which $\partial H/\partial u \equiv 0$ or $\partial H/\partial v \equiv 0$ are called singular arcs. In (Ref [1]), Anderson presents the characteristics of singular solutions in two-person, zero sum differential games including a set of necessary conditions for the optimality of the solution. Defining $S_p \equiv \partial H/\partial u$, necessary conditions for the existence of a singular solution are that S_p $(\underline{x}, \underline{\lambda})$ and all time derivatives of S_p must vanish. Successive differentiation of S_p generally results in an equation which explicitly contains u which allows the determination of the singular control u_s. Similar statements apply for the evader when $S_e \equiv \partial H/\partial v = 0$. The necessary conditions for the singular control u_s to minimize J and for v_s to maximize J are

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$$(-1)^q \frac{\partial}{\partial u} \left[\frac{d^{2q} S_p}{dt^{2q}} \right] \geq 0$$

and (10)

$$(-1)^q \frac{\partial}{\partial v} \left[\frac{d^{2q} S_e}{dt^{2q}} \right] \leq 0$$

where $2q$ is the order of the derivative which explicitly contains u and v. No sufficient conditions for optimality cf singular solutions are available.

It was previously stated that the optimal solution to the differential game problem is the pair of controls (u^*, v^*) which provide a saddle point of J. If the pair (u^*, v^*) is given as ($u^*(t)$, $v^*(t)$), one speaks of an open-loop solution. If the controls are expressed as functions of the instantaneous state and time

$$u^* = k_u (\underline{x}, t)$$

$$v^* = k_v (\underline{x}, t)$$

one has what is known as a feedback or closed-loop control law.

The importance of the difference between open and closed-loop controls in differential games can be made clear by returning to the inequalities of Eq (3). The second inequality can be considered in two ways depending on how the minimizing player considers the maximizing player's controls to be expressed. We can have either

$$\min_u J [u, v^*(t)] = J [u^*(t; \underline{x}_o, t_o); v^*(t)]$$

or

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$$\min_{v} J\,[u,\,k_v\,(\underline{x},\,t)] = J\,[u^*\,(t;\,\underline{x}_o,\,t_o\,);\,k_v\,(\underline{x},\,t)]$$

The latter requires that u^* be optimal against an opponent whose control is produced in a closed-loop manner implying that v can immediately take advantage of any non-optimal play made by u. It is obvious, therefore, that in differential game problems feedback strategies must be considered. If the open-loop problem can be solved analytically, the open-loop control can be computed instantaneously and continuously for any initial state and hence generate an optimal closed-loop control.

If one assumes the existence of strategies

$$u^* = k_u\,(\underline{x},\,t) \text{ and } v^* = k_v\,(\underline{x},\,t)$$

that provide a saddle point of J, the following significance to Eq (3) can be given. If the maximizing player uses his optimal strategy, he is guaranteed a payoff at least equal to the value $J\,(u^*,\,v^*)$ and if the minimizing player selects his optimal strategy, he guarantees that his opponent will get a payoff no greater than the value.

Sufficient Conditions

In the previous section, it was pointed out that solutions to the two point boundary value problem given by Eq (6), (8), and (9), do not necessarily satisfy the saddle point conditions

$$J\,(u^*,\,v) \leq J\,(u^*,\,v^*) \leq J\,(u,\,v^*)$$

Verification of the saddle point conditions can be accomplished directly by verifying the inequalities of Eq (3) separately through the consideration of two optimal control problems. The first

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inequality can be verified by solving the problem

$$J(u^*, v^*) = \min_{u} J(u, v^*)$$

where v^* may be given in open or closed-loop form. The second
inequality is verified by solving the problem

$$J(u^*, v^*) = \max_{v} J(u^*, v)$$

where u^* may be expressed in open or closed-loop form. The saddle
point is established if u^* and v^* from both problems are the same.

Sufficient conditions for a saddle point solution for this differ-
ential game problem are discussed in Appendix A. When u^* and v^* are
interior controls the following conditions are sufficient for (u^*, v^*)
to provide a saddle point of J.

(1) The strengthened Legendre-Clebsch conditions:

$$H_{uu} > 0 \qquad\qquad H_{vv} < 0$$

(2) The non-existence of a conjugate point on $[t_o, t_f)$
for an accessory minimax problem.

IV. PURSUIT-EVASION DIFFERENTIAL GAME - STANDARD AIRCRAFT MODEL

A differential game problem using a realistic aircraft model was discussed in Chapter II. The purpose of this chapter is to obtain open-loop solutions to this differential game problem using the necessary conditions given in Chapter III.

Statement of the Problem

The problem is to determine a saddle point of

$$J\,(t_f) = R(t_f) = [(x_e - x_p)^2 + (h_e - h_p)^2]^{1/2}_{t=t_f}$$

subject to

$$\dot{x}_p = V_{xp}$$

$$\dot{h}_p = V_{hp} \qquad\qquad\qquad (11)$$

$$\dot{V}_{xp} = D_p \;[(C_{Tp} - k_p C_{Lp}^2)\cos\gamma_p - C_{Lp}\sin\gamma_p]$$

$$\dot{V}_{hp} = D_p \;[(C_{Tp} - k_p C_{Lp}^2)\sin\gamma_p + C_{Lp}\cos\gamma_p] - G$$

$$\dot{x}_e = V_{xe}$$

$$\dot{h}_e = V_{he} \qquad\qquad\qquad (12)$$

$$\dot{V}_{xe} = D_e \;[(C_{Te} - k_e C_{Le}^2)\cos\gamma_e - C_{Le}\sin\gamma_e]$$

$$\dot{V}_{he} = D_e \;[(C_{Te} - k_e C_{Le}^2)\sin\gamma_e + C_{Le}\cos\gamma_e] - G$$

where

$$\gamma_p \equiv \tan^{-1}\,(V_{hp}/V_{xp})$$

$$\gamma_e \equiv \tan^{-1}(V_{he}/V_{xe})$$

and the subscripts p and e refer to the pursuer and evader, respectively. The final time (t_f) as well as the initial values of the state variables are to be considered specified. C_{Lp} and C_{Le} are the controls for the pursuer and evader, respectively. They are bounded by

$$C_{Lp\ min} \leq C_{Lp} \leq C_{Lp\ max}$$

$$C_{Le\ min} \leq C_{Le} \leq C_{Le\ max}$$

(13)

For convenience D_p and D_e are substituted for $(Q\ S/m)_p$ and $(Q\ S/m)_e$. D_p, C_{Tp}, D_e C_{Te} are positive constants.

Necessary Conditions

Applying necessary conditions for a saddle point solution, the Hamiltonian is

$$H = \lambda_{xe} V_{xe} + \lambda_{he} V_{he} + \lambda_{Vxe} D_e \left[(C_{Te} - k_e C_{Le}^2) \cos \gamma_e \right.$$

$$- C_{Le} \sin \gamma_e] + \lambda_{Vhe} D_e \left[(C_{Te} - k_e C_{Le}^2) \sin \gamma_e \right.$$

$$+ C_{Le} \cos \gamma_e] - \lambda_{Vhe} G + \lambda_{xp} V_{xp} + \lambda_{hp} V_{hp}$$

$$+ \lambda_{Vxp} D_p \left[(C_{Tp} - k_p C_{Lp}^2) \cos \gamma_p - C_{Lp} \sin \gamma_p \right]$$

$$+ \lambda_{Vhp} D_p \left[(C_{Tp} - k_p C_{Lp}^2) \sin \gamma_p + C_{Lp} \cos \gamma_p \right] - \lambda_{Vhp} G$$

The Hamiltonian is to be minimized with respect to C_{Lp} and maximized with respect to C_{Le} subject to the constraints given by Eq (13). If the minimizing and maximizing controls are on the interior of their admissible sets, it is necessary that

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$$\partial H / \partial C_{Lp} = 0 \qquad\qquad \partial H / \partial C_{Le} = 0$$

and

$$\partial^2 H / \partial C_{Lp}^{\ 2} \geq 0 \qquad\qquad \partial^2 H / \partial C_{Le}^{\ 2} \leq 0$$

Solving $\partial H / \partial C_{Lp} = 0$, the following expression for $\bar{C}_{Lp}$ is obtained.

$$\bar{C}_{Lp} = \tan\,(\beta_p - \gamma_p) \,/\, (2k_p)$$

where

$$\beta_p \equiv \tan^{-1}\,(\lambda_{Vhp} \,/\, \lambda_{Vxp})$$

Similarly for e, $\partial H / \partial C_{Le} = 0$ yields

$$\bar{C}_{Le} = \tan\,(\beta_e - \gamma_e) \,/\, (2k_e)$$

where

$$\beta_e \equiv \tan^{-1}\,(\lambda_{Vhe} \,/\, \lambda_{Vxe})$$

$\bar{C}_{Lp}$ and $\bar{C}_{Le}$ are minimizing and maximizing controls respectively, provided that

$$\partial^2 H / \partial C_{Lp}^{\ 2} = -\,2k_p\,D_p\,[\sin\,\gamma_p\,\lambda_{Vhp} + \cos\,\gamma_p\,\lambda_{Vxp}] \geq 0$$

and   (14)

$$\partial^2 H / \partial C_{Le}^{\ 2} = -\,2k_e\,D_e\,[\sin\,\gamma_e\,\lambda_{Vhe} + \cos\,\gamma_e\,\lambda_{Vxe}] \leq 0$$

The control C_{Lp}^{*} that minimizes H subject to Eq (13) is

$$C_{Lp}^{*} = \begin{cases} C_{Lp\,max} & \bar{C}_{Lp} > C_{Lp\,max} \\[2mm] \bar{C}_{Lp} & C_{Lp\,min} \leq \bar{C}_{Lp} \leq C_{Lp\,max} \\[2mm] C_{Lp\,min} & \bar{C}_{Lp} \leq C_{Lp\,min} \end{cases} \qquad (15)$$

The control C_{Le}^* that maximizes H subject to Eq (13) is

$$C_{Le}^* = \begin{bmatrix} C_{Le\ max} & \bar{C}_{Le} > C_{Le\ max} \\ \\ \bar{C}_{Le} & C_{Le\ min} \leq \bar{C}_{Le} \leq C_{Le\ max} \\ \\ C_{Le\ min} & \bar{C}_{Le} < C_{Le\ min} \end{bmatrix} \qquad (16)$$

The Euler-Lagrange or costate equations are

$$\dot{\lambda}_{xe} = 0$$

$$\dot{\lambda}_{he} = 0$$

$$\dot{\lambda}_{Vxe} = -\lambda_{xe} - \lambda_{Vxe}\, D_e\, [(C_{Te} - k_e\, C_{Le}^2)\, \partial(\cos \gamma_e)/\partial V_{xe}$$

$$-C_{Le}\, \partial(\sin \gamma_e)/\partial V_{xe}] - \lambda_{Vhe}\, D_e\, [(C_{Te} \qquad (17)$$

$$-k_e\, C_{Le}^2)\, \partial(\sin \gamma_e)/\partial V_{xe} + C_{Le}\, \partial(\cos \gamma_e)/\partial V_{xe}]$$

$$\dot{\lambda}_{Vhe} = -\lambda_{he} - \lambda_{Vxe}\, D_e\, [(C_{Te} - k_e\, C_{Le}^2)\, \partial(\cos \gamma_e)/\partial V_{he}$$

$$-C_{Le}\, \partial(\sin \gamma_e)/\partial V_{he}] - \lambda_{Vhe}\, D_e\, [(C_{Te}$$

$$-k_e\, C_{Le}^2)\, \partial(\sin \gamma_e)/\partial V_{he} + C_{Le}\, \partial(\cos \gamma_e)/\partial V_{he}]$$

where

$$\partial(\cos \gamma_e)/\partial V_{xe} = \sin^2 \gamma_e /[V_{xe}^2 + V_{he}^2]^{1/2}$$

$$\partial(\sin \gamma_e)/\partial V_{xe} = - \sin \gamma_e \cos \gamma_e /[V_{xe}^2 + V_{he}^2]^{1/2}$$

$$\partial(\cos \gamma_e)/\partial V_{he} = - \sin \gamma_e \cos \gamma_e /[V_{xe}^2 + V_{he}^2]^{1/2} \qquad (18)$$

$$\partial(\sin \gamma_e)/\partial V_{he} = \cos^2 \gamma_e /[V_{xe}^2 + V_{he}^2]^{1/2}$$

$$\dot{\lambda}_{xp} = 0$$

$$\dot{\lambda}_{hp} = 0$$

$$\dot{\lambda}_{Vxp} = -\lambda_{xp} - \lambda_{Vxp} D_p \left[(C_{Tp} - k_p C_{Lp}^2) \, \partial(\cos \gamma_p)/\partial V_{xp} \right.$$

$$-C_{Lp} \, \partial(\sin \gamma_p)/\partial V_{xp}] - \lambda_{Vhp} D_p [(C_{Tp}$$

$$-k_p C_{Lp}^2) \, \partial(\sin \gamma_p)/\partial V_{xp} + C_{Lp} \, \partial(\cos \gamma_p)/\partial V_{xp}] \qquad (19)$$

$$\dot{\lambda}_{Vhp} = -\lambda_{hp} - \lambda_{Vxp} D_p \left[(C_{Tp} - k_p C_{Lp}^2) \, \partial(\cos \gamma_p)/\partial V_{hp} \right.$$

$$-C_{Lp} \, \partial(\sin \gamma_p)/\partial V_{hp}] - \lambda_{Vhp} D_p [(C_{Tp}$$

$$-k_p C_{Lp}^2) \, \partial(\sin \gamma_p)/\partial V_{hp} + C_{Lp} \, \partial(\cos \gamma_p)/\partial V_{hp}]$$

where the partial derivatives are defined as in Eq (18) with the subscript e replaced by p.

The transversality conditions defined by $\underline{\lambda}^T (t_f) = R_{\underline{x}|_{t_f}}$ are

$$\lambda_{xe} (t_f) = -\lambda_{xp} (t_f) = (x_e - x_p)/R \mid_{t = t_f}$$

$$\lambda_{he} (t_f) = -\lambda_{hp} (t_f) = (h_e - h_p)/R \mid_{t = t_f}$$

$$\lambda_{Vxe} (t_f) = 0$$

$$\lambda_{Vhe} (t_f) = 0 \qquad (20)$$

$$\lambda_{Vxp} (t_f) = 0$$

$$\lambda_{Vhp} (t_f) = 0$$

Problem Solution

The problem stated above is a two-point boundary value problem

(TPBVP). Because of the non-linearity of the problem, no closed form solution is known. Open-loop solutions can be obtained only through the use of numerical iterative techniques.

Trajectories satisfying the necessary conditions for a saddle point solution to this problem may be generated by assuming terminal state conditions

$$\underline{x} \ (t_f) = \underline{x}_{t_f}$$

and simultaneously integrating backwards in time the state differential equations given by Eqs (11) and (12) and the costate differential equations given by Eqs (17) and (19) with conditions at t_f given by Eq (20). The saddle point controls, C_{Lp}^{*} and C_{Le}^{*} are given by Eqs (14) and (15). These trajectories represent open-loop solutions to the differential game problem where the specified initial state conditions $\underline{x} \ (t_o)$ correspond to a point $\underline{x} \ (t)$ on these trajectories and the fixed flight time of the game is equal to $t_f - t$.

Figures 4 and 5 show two solutions obtained in this manner. The data used to obtain the solution shown in Figure 4 is

$$x_e \ (t_f) = 33383 \text{ feet} \qquad x_p \ (t_f) = 28262 \text{ feet}$$

$$h_e \ (t_f) = 18808 \text{ feet} \qquad h_p \ (t_f) = 27278 \text{ feet}$$

$$V_{xe} \ (t_f) = 603 \text{ ft/sec} \qquad V_{xp} \ (t_f) = 740 \text{ ft/sec}$$

$$V_{he} \ (t_f) = -996 \text{ ft/sec} \qquad V_{hp} \ (t_f) = -1224 \text{ ft/sec}$$

$$D_e = 122 \text{ ft/sec}^2 \qquad D_p = 192 \text{ ft/sec}^2$$

$$C_{Te} = .06 \qquad C_{Tp} = .04$$

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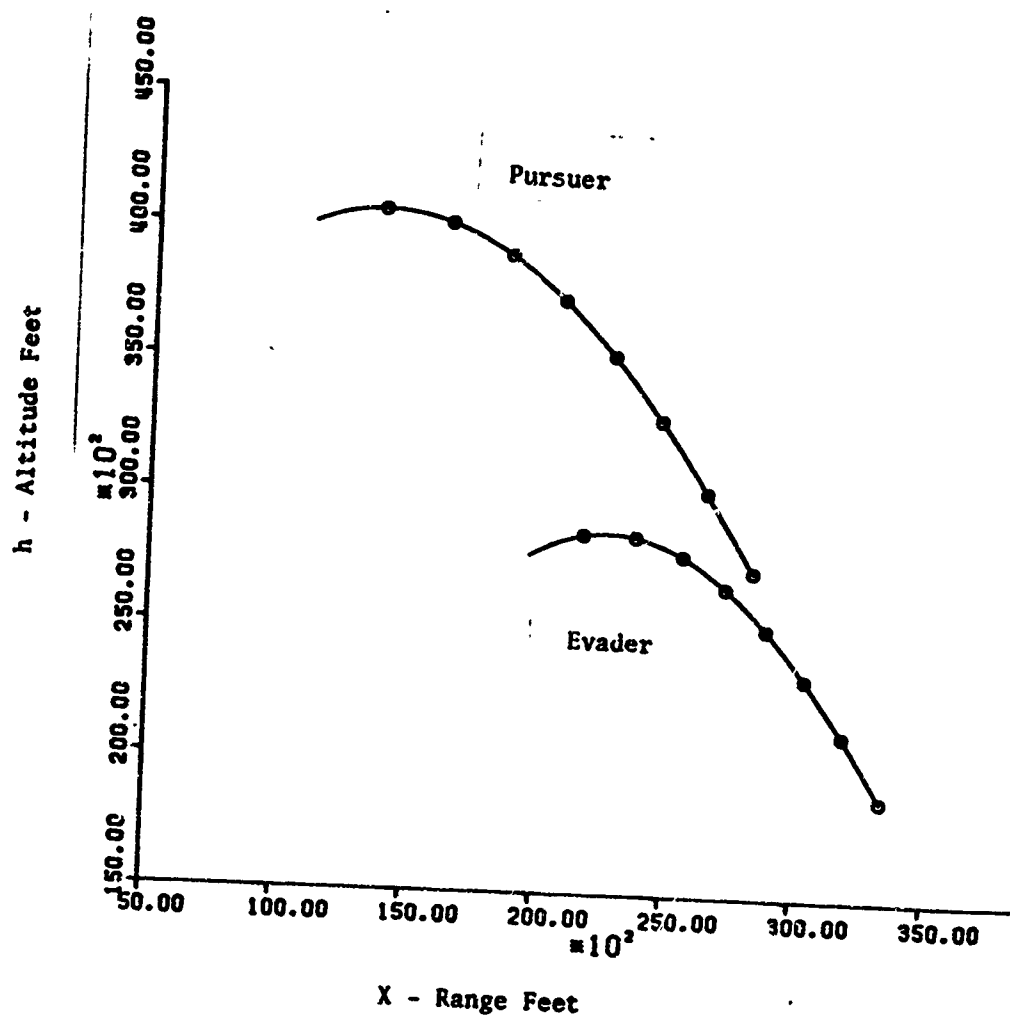
Figure 4. Optimal Pursuit-Evasion Trajectories - Standard
Aircraft Model
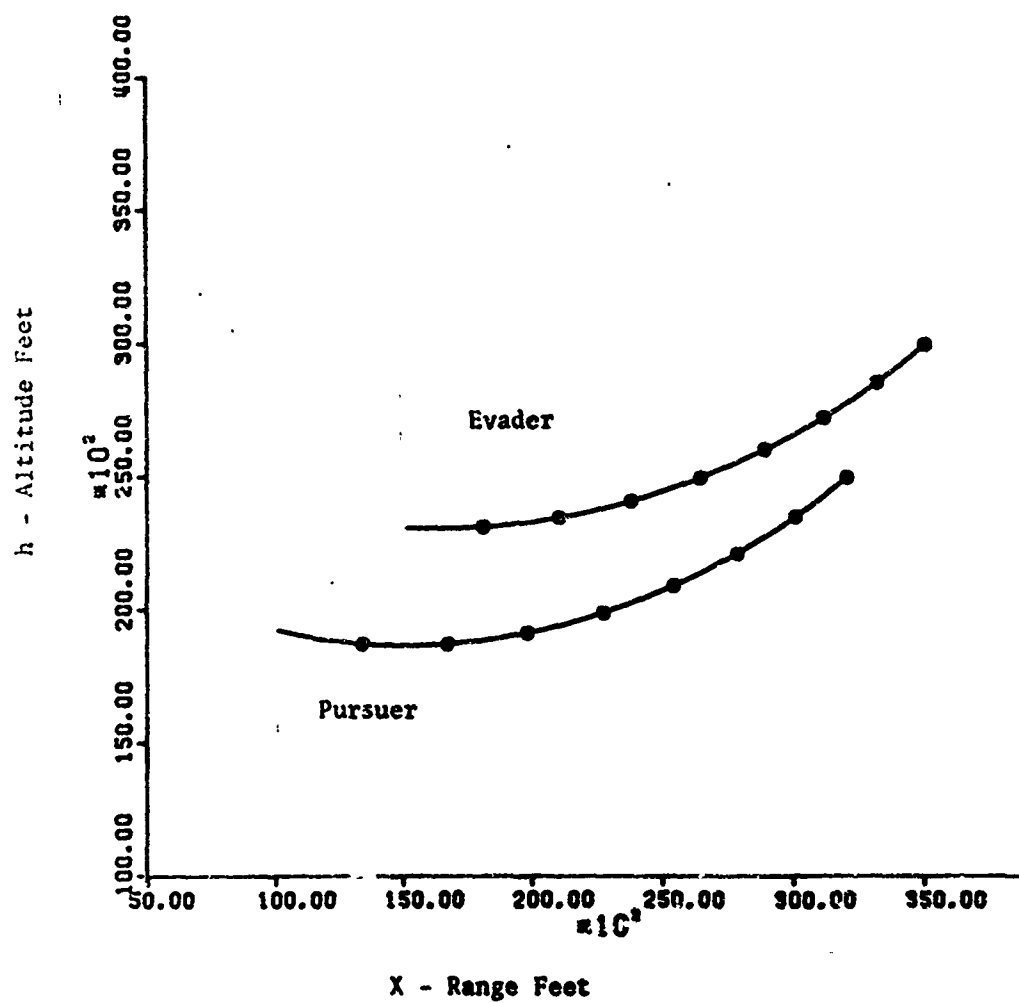
Figure 5. Optimal Pursuit-Evasion Trajectories - Standard
Aircraft Model

$$k_e = .5 \qquad\qquad k_p = .5$$

$$|C_{Le}| \le 1 \qquad\qquad |C_{Lp}| \le 1$$

The trajectories in Figure 4 can be viewed as the solution to the differential game problem where the initial position of the pursuer is above and behind the evader.

The data used to obtain the solution shown in Figure 5 is

$$x_e(t_f) = 35000 \text{ feet} \qquad x_p(t_f) = 32000 \text{ feet}$$

$$h_e(t_f) = 30000 \text{ feet} \qquad h_p(t_f) = 25000 \text{ feet}$$

$$V_{xe}(t_f) = 688 \text{ ft/sec} \qquad V_{xp}(t_f) = 727 \text{ ft/sec}$$

$$V_{he}(t_f) = 580 \text{ ft/sec} \qquad V_{hp}(t_f) = 612 \text{ ft/sec}$$

$$D_e = 160 \text{ ft/sec}^2 \qquad D_p = 175 \text{ ft/sec}^2$$

$$C_{Te} = .06 \qquad\qquad C_{Tp} = .04$$

$$k_e = .5 \qquad\qquad k_p = .5$$

$$|C_{Le}| \le 1 \qquad\qquad |C_{Lp}| \le 1$$

In the differential game problem corresponding to this solution, the pursuer is initially behind and below the evader.

These solutions will be used as a comparative base for the solutions obtained using the simplified aircraft models which are analyzed in the next three chapters.

V. PURSUIT-EVASION DIFFERENTIAL GAME - STATIC MODEL

In Chapter II, the pursuit-evasion differential game problem was cast in terms of the acceleration vectogram associated with the aircraft model. The difficulty in solving the differential game with the standard aircraft model resulted from the parabolic shape of the vectogram and its orientation with respect to the changing aircraft flight path direction.

In this chapter the problem is simplified by assuming that the orientation of the vectogram and the aircraft flight path direction are fixed. This is equivalent to a static aircraft analysis with a constant flight path angle. The acceleration vectogram for this model is the same as Figure 3, except that the flight path inclination angle is fixed.

The purpose of this chapter is to solve the pursuit-evasion differential game problem with the static aircraft model and to discuss the application of the resulting closed-loop control law to the standard aircraft model.

Statement of the Problem

The problem is to determine a saddle point of

$$J(t_f) = R(t_f) = [(x_e - x_p)^2 + (h_e - h_p)^2]^{1/2}_{t=t_f}$$

subject to

$$\dot{x}_p = V_{xp}$$

$$\dot{h}_p = V_{hp} \tag{21}$$

$$\dot{V}_{xp} = D_p [(C_{Tp} - k_p C_{Lp}^2) \cos \xi - C_{Lp} \sin \xi]$$

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$$\dot{V}_{hp} = D_p [(C_{Tp} - k_p C_{Lp}^2) \sin \xi + C_{Lp} \cos \xi] - G$$

$$\dot{x}_e = V_{xe}$$

$$\dot{h}_e = V_{he} \tag{22}$$

$$\dot{V}_{xe} = D_e [(C_{Te} - k_e C_{Le}^2) \cos \psi - C_{Le} \sin \psi]$$

$$\dot{V}_{he} = D_e [(C_{Te} - k_e C_{Le}^2) \sin \psi + C_{Le} \cos \psi] - G$$

where the subscripts p and e refer to the pursuer and evader, respectively. ξ is the specified constant flight path angle for the pursuer and ψ is the specified constant flight path angle for the evader. The final time (t_f) as well as the initial values of the state variables are to be considered specified. C_{Lp} and C_{Le} are the control variables for the pursuer and evader, respectively. They are bounded by

$$C_{Lp\ min} \leq C_{Lp} \leq C_{Lp\ max}$$

$$C_{Le\ min} \leq C_{Le} \leq C_{Le\ max} \tag{23}$$

D_p, C_{Tp}, D_e, C_{Te} are positive constants defined as in Chapter IV.

Necessary Conditions

Applying necessary conditions for a saddle point solution, the Hamiltonian is

$$H = \lambda_{xe} V_{xe} + \lambda_{he} V_{he} + \lambda_{Vxe} D_e [(C_{Te} - k_e C_{Le}^2) \cos \psi$$

$$- C_{Le} \sin \psi] + \lambda_{Vhe} D_e [(C_{Te} - k_e C_{Le}^2) \sin \psi$$

$$+ C_{Le} \cos \psi] - \lambda_{Vhe} G + \lambda_{xp} V_{xp} + \lambda_{hp} V_{hp} +$$

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$$\lambda_{Vxp} D_p [(C_{Tp} - k_p C_{Lp}^2) \cos \xi - C_{Lp} \sin \xi]$$

$$+ \lambda_{Vhp} D_p [(C_{Yp} - k_p C_{Lp}^2) \sin \xi + C_{Lp} \cos \xi] - \lambda_{Vhp} G$$

The Hamiltonian is to be minimized with respect to C_{Lp} and maximized with respect to C_{Le} subject to the constraints given by Eq (23). If the minimizing and maximizing controls are on the interior of their admissible sets, it is necessary that

$$\partial H/\partial C_{Lp} = 0 \qquad\qquad \partial H/\partial C_{Le} = 0$$

and

$$\partial^2 H/\partial C_{Lp}^2 \geq 0 \qquad\qquad \partial^2 H/\partial C_{Le}^2 \leq 0$$

Solving $\partial H/\partial C_{Lp} = 0$, one obtains

$$\bar{C}_{Lp} = \tan (\theta_p - \xi)/(2k_p) \tag{24}$$

where

$$\theta_p \equiv \tan^{-1} (\lambda_{Vhp}/\lambda_{Vxp}) \tag{25}$$

Similarly for the evader, $\partial H/\partial C_{Le} = 0$ yields

$$\bar{C}_{Le} = \tan (\theta_e - \psi)/(2k_e) \tag{26}$$

where

$$\theta_e \equiv \tan^{-1} (\lambda_{Vhs}/\lambda_{Vxe}) \tag{27}$$

$\bar{C}_{Lp}$ and $\bar{C}_{Le}$ are minimizing and maximizing controls respectively, provided that

$$\partial^2 H/\partial C_{Lp}^2 = - 2k_p D_p [\sin \xi \, \lambda_{Vhp} + \cos \xi \, \lambda_{Vxp}] \geq 0$$

and

$$\partial^2 H/\partial C_{Le}^2 = -2k_e \dot{D}_e [\sin \psi \, \lambda_{Vhe} + \cos \psi \, \lambda_{Vxe}] \le 0$$

The control C_{Lp}^* that minimizes H subject to Eq (23) is

$$
C_{Lp}^* =
\begin{cases}
C_{Lp \; max} & \bar{C}_{Lp} > C_{Lp \; max} \\
\bar{C}_{Lp} & C_{Lp \; min} \le \bar{C}_{Lp} \le C_{Lp \; max} \\
C_{Lp \; min} & \bar{C}_{Lp} \le C_{Lp \; min}
\end{cases}
\tag{28}
$$

and the control C_{Le}^* that maximizes H subject to Eq (23) is

$$
C_{Le}^* =
\begin{cases}
C_{Le \; max} & \bar{C}_{Le} > C_{Le \; max} \\
\bar{C}_{Le} & C_{Le \; min} \le \bar{C}_{Le} \le C_{Le \; max} \\
C_{Le \; min} & \bar{C}_{Le} < C_{Le \; min}
\end{cases}
\tag{29}
$$

The costate equations are

$$\dot{\lambda}_{xe} = 0$$

$$\dot{\lambda}_{he} = 0$$

$$\dot{\lambda}_{Vxe} = -\lambda_{xe}$$

$$\dot{\lambda}_{Vhe} = -\lambda_{he}$$

$$\dot{\lambda}_{xp} = 0$$

$$\dot{\lambda}_{hp} = 0$$

$$\dot{\lambda}_{Vxp} = -\lambda_{xp}$$

$$\dot{\lambda}_{Vhp} = -\lambda_{hp}$$

The transversality conditions are

$$\lambda_{xe}(t_f) = -\lambda_{xp}(t_f) = (x_e - x_p)/R \big|_{t=t_f}$$

$$\lambda_{he}(t_f) = -\lambda_{hp}(t_f) = (h_e - h_p)/R \big|_{t=t_f}$$

$$\lambda_{Vxe}(t_f) = 0$$

$$\lambda_{Vhe}(t_f) = 0$$

$$\lambda_{Vxp}(t_f) = 0$$

$$\lambda_{Vhp}(t_f) = 0$$

Problem Solution

Integrating the costate equations and applying the transversality conditions yields

$$\lambda_{Vxp}(t) = -\lambda_{xp}(t_f)(t_f - t)$$

$$\lambda_{Vhp}(t) = -\lambda_{hp}(t_f)(t_f - t) \qquad (30)$$

$$\lambda_{Vxe}(t) = -\lambda_{xe}(t_f)(t_f - t)$$

$$\lambda_{Vhe}(t) = -\lambda_{he}(t_f)(t_f - t)$$

From Eqs (30) and the transversality conditions

$$[\lambda_{Vhp}(t)/\lambda_{Vxp}(t)] = \lambda_{hp}(t_f)/\lambda_{xp}(t_f)$$

$$= (h_e - h_p)/(x_e - x_p) \big|_{t=t_f}$$

and

$$[\lambda_{Vhe}(t)/\lambda_{Vxe}(t)] = \lambda_{he}(t_f)/\lambda_{xe}(t_f) =$$

$$(h_e - h_p)/(x_e - x_p)\ |_{t=t_f}$$

Comparing these equations with Eqs (25) and (27) shows that

$$\theta_e\ (t) = \theta_p\ (t) = \theta\ (t_f)$$

where θ is the angle measured from the horizontal to the line of sight between the pursuer and evader at termination. Since θ_e and θ_p are constant functions, the controls C_{Lp}^* and C_{Le}^* are constants from Eqs (24), (26), (28) and (29).

Integrating the state equations $\dot{V}_{xe}$ and $\dot{V}_{xp}$ and combining the results yields

$$V_{xe}\ (t) - V_{xp}\ (t) = V_{xe}\ (t_o) - V_{xp}\ (t_o) \tag{31}$$

$$+ \{D_e\ [(C_{Te} - k_e\ C_{Le}^{*\ 2})\ \cos \psi - C_{Le}^*\ \sin \psi] - D_p\ [(C_{Tp}$$

$$- k_p\ C_{Lp}^{*\ 2})\ \cos \xi - C_{Lp}^*\ \sin \xi]\}(t - t_o)$$

Integrating Eq (31) yields

$$x_e\ (t) - x_p\ (t) = x_e\ (t_o) - x_p\ (t_o) + [V_{xe}\ (t_o) - V_{xp}\ (t_o)]\ (t - t_o)$$

$$+ \{D_e\ [(C_{Te} - k_e\ C_{Le}^{*\ 2})\ \cos \psi - C_{Le}^*\ \sin \psi] \tag{32}$$

$$- D_p\ [(C_{Tp} - k_p\ C_{Lp}^{*\ 2})\ \cos \xi - C_{Lp}^*\ \sin \xi]\}(t - t_o)^2/_2$$

The following equation is similarly obtained for the h components of the state equations

$$h_e\ (t) - h_p\ (t) = h_e\ (t_o) - h_p\ (t_o) + [V_{he}\ (t_o) - V_{hp}\ (t_o)](t - t_o)$$

$$+ \{D_e\ [(C_{Te} - k_e\ C_{Le}^{*\ 2})\ \sin \psi + C_{Le}^*\ \cos \psi] \tag{33}$$

$$- D_p\ [(C_{Tp} - k_p\ C_{Lp}^{*\ 2})\ \sin \xi + C_{Lp}^*\ \cos \xi]\}(t - t_o)^2/_2$$

Dividing Eq (33) by Eq (32) and evaluating at $t = t_f$ yields

$$\tan \theta = \left\{ h_e(t_o) - h_p(t_o) + [V_{he}(t_o) - V_{hp}(t_o)](t_f - t_o) \right.$$
$$+ \{D_e [(C_{Te} - k_e C_{Le}^{*2}) \sin \psi + C_{Le}^* \cos \psi]$$
$$- D_p [(C_{Tp} - k_p C_{Lp}^{*2}) \sin \xi$$
$$\left. + C_{Lp}^* \cos \xi]\}(t_f - t_o)^2/2 \right\} \bigg/ \left\{ x_e(t_o) - x_p(t_o) \right. \qquad (34)$$
$$+ [V_{xe}(t_o) - V_{xp}(t_o)](t_f - t_o)$$
$$+ \{D_e [(C_{Te} - k_e C_{Le}^{*2}) \cos \psi - C_{Le}^* \sin \psi]$$
$$\left. - D_p [(C_{Tp} - k_p C_{Lp}^{*2}) \cos \xi - C_{Lp}^* \sin \xi]\}(t_f - t_o)^2/2 \right\}$$

Eq (34) is a transcendental equation with the unknown parameter θ since C_{Lp}^* and C_{Le}^* are functions of θ only and all state components are specified at t_o. This equation can be solved for θ using a digital computer. The determination of θ allows C_{Lp}^* and C_{Le}^* to be computed and therefore effectively completes the solution to this differential game problem. The ability to continuously determine the optimal controls as a function of the present state of the problem and the time to go constitutes a closed-loop control law for this differential game problem.

Pseudo-Dynamic Application

The differential game problem solved in this chapter is based on a fixed orientation of the aircraft flight path. A possible application of this solution to the standard model problem of the previous chapter would be to solve the static problem at each integration or specified time step as if the flight path angles of the aircraft were

to remain fixed. After determining the controls C_{Lp} and C_{Le} in this manner, they would be applied to the standard model dynamics. This procedure would be a closed-loop control law for the standard model problem. Results of the application of this procedure are presented in Chapter VIII.

VI.   PURSUIT-EVASION DIFFERENTIAL GAME - ZERO INDUCED DRAG MODEL

In this chapter another aircraft model is considered. This model is characterized by the simplification of neglecting drag due to lift (induced drag). This simplification is accomplished mathematically by assigning the value zero to the induced drag factors k_p and k_e in Eqs (11) and (12). A further simplification is introduced by neglecting gravitational forces. The improvement of this model over the static model is that the flight path orientation is not fixed ($\dot{\gamma} \neq 0$). The acceleration vectogram representing this model is shown in Figure 6 and is termed the zero induced drag model.
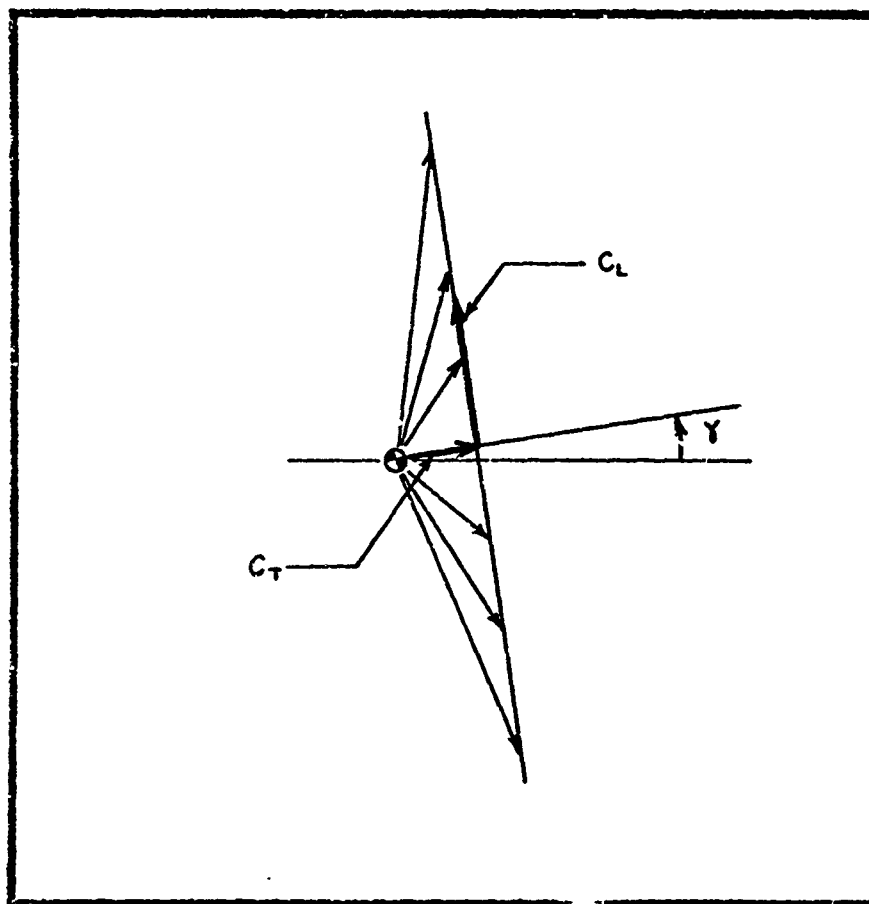
Figure 6.   Acceleration Vectogram for the Zero Induced Drag Aircraft
Model

The purpose of this chapter is to solve the pursuit-evasion differential game with the aircraft modeled as above.

Statement of the Problem

The problem is to determine a saddle point of

$$J\ (t_f) = R\ (t_f) = [(x_e - x_p)^2 + (h_e - h_p)^2]^{1/2}_{t=t_f}$$

subject to the state equations

$$\dot{x}_p = V_{xp}$$

$$\dot{h}_p = V_{hp} \qquad\qquad (35)$$

$$\dot{V}_{xp} = D_p\ [C_{Tp}\ \cos\ \gamma_p - C_{Lp}\ \sin\ \gamma_p]$$

$$\dot{V}_{hp} = D_p\ [C_{Tp}\ \sin\ \gamma_p + C_{Lp}\ \cos\ \gamma_p]$$

$$\dot{x}_e = V_{xe}$$

$$\dot{h}_e = V_{he} \qquad\qquad (36)$$

$$\dot{V}_{xe} = D_e\ [C_{Te}\ \cos\ \gamma_e - C_{Le}\ \sin\ \gamma_e]$$

$$\dot{V}_{he} = D_e\ [C_{Te}\ \sin\ \gamma_e + C_{Le}\ \cos\ \gamma_e]$$

where

$$\gamma_p \equiv \tan^{-1}\ (V_{hp}/V_{xp})$$

$$\gamma_e \equiv \tan^{-1}\ (V_{he}/V_{xe})$$

and the subscripts p and e refer to the pursuer and evader, respectively. The final time (t_f) as well as the initial values of the state variables are specified. D_p, C_{Tp}, D_e, C_{Te} are constants defined in

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Chapter IV. C_{Lp} and C_{Le} are the controls for the pursuer and evader, respectively. They are bounded by

$$C_{Lp\ min} \leq C_{Lp} \leq C_{Lp\ max}$$

$$C_{Le\ min} \leq C_{Le} \leq C_{Le\ max}$$

Analysis of this problem is made easier if the state variables V_x and V_h are replaced by V and γ where

$$V = [V_x^2 + V_h^2]^{1/2} \quad\text{and}\quad \gamma = \tan^{-1}(V_h/V_x)$$

The resulting state equations are

$$\dot{x}_p = V_p \cos \gamma_p$$

$$\dot{h}_p = V_p \sin \gamma_p$$

$$\dot{V}_p = D_p C_{Tp} \tag{37}$$

$$\dot{\gamma}_p = D_p C_{Lp}/V_p$$

$$\dot{x}_e = V_e \cos \gamma_e$$

$$\dot{h}_e = V_e \sin \gamma_e$$

$$\dot{V}_e = D_e C_{Te} \tag{38}$$

$$\dot{\gamma}_e = D_e C_{Le}/V_e$$

which replace Eqs (35) and (36).

Necessary Conditions

Applying necessary conditions for a saddle point solution, the Hamiltonian is

$$H = \lambda_{xe} V_e \cos \gamma_e + \lambda_{he} V_e \sin \gamma_e + \lambda_{Ve} D_e C_{Te}$$
$$+ \lambda_{\gamma e} D_e C_{Le}/V_e + \lambda_{xp} V_p \cos \gamma_p + \lambda_{hp} V_p \sin \gamma_p$$
$$+ \lambda_{Vp} D_p C_{Tp} + \lambda_{\gamma p} D_p C_{Lp}/V_p$$

The costate equations are

$$\dot{\lambda}_{xp} = 0$$

$$\dot{\lambda}_{hp} = 0$$

$$\dot{\lambda}_{Vp} = - \lambda_{xp} \cos \gamma_p - \lambda_{hp} \sin \gamma_p + \lambda_{\gamma p} D_p C_{Lp}/V_p^2$$

$$\dot{\lambda}_{\gamma p} = V_p (\lambda_{xp} \sin \gamma_p - \lambda_{hp} \cos \gamma_p)$$

$$\dot{\lambda}_{xe} = 0$$

$$\dot{\lambda}_{he} = 0$$

$$\dot{\lambda}_{Ve} = - \lambda_{xe} \cos \gamma_e - \lambda_{he} \sin \gamma_e + \lambda_{\gamma e} D_e C_{Le}/V_e^2$$

$$\dot{\lambda}_{\gamma e} = V_e (\lambda_{xe} \sin \gamma_e - \lambda_{he} \cos \gamma_e)$$

The transversality conditions yield

$$\lambda_{xe} (t_f) = - \lambda_{xp} (t_f) = (x_e - x_p)/R \mid_{t=t_f}$$

$$\lambda_{he} (t_f) = - \lambda_{hp} (t_f) = (h_e - h_p)/R \mid_{t=t_f}$$

$$\lambda_{Ve} (t_f) = 0$$

$$\lambda_{\gamma e} (t_f) = 0$$

$$\lambda_{Vp} (t_f) = 0$$

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$$\lambda_{\gamma_p}(t_f) = 0$$

Minimizing the Hamiltonian with respect to C_{Lp} and maximizing with respect to C_{Le} subject to the constraints on C_{Lp} and C_{Le} defines the controls

$$\bar{C}_{Le} = \begin{cases} C_{Le\ max} & ; & S_e > 0 \\ \\ C_{Le\ min} & ; & S_e < 0 \end{cases}$$

$$\bar{C}_{Lp} = \begin{cases} C_{Lp\ max} & ; & S_p < 0 \\ \\ C_{Lp\ min} & ; & S_p > 0 \end{cases}$$

(39)

where

$$S_e \equiv \partial H/\partial C_{Le} = \lambda_{\gamma_e} D_e/V_e$$

$$S_p \equiv \partial H/\partial C_{Lp} = \lambda_{\gamma_p} D_p/V_p$$

However, if S_e is equal to zero, the Hamiltonian is independent of C_{Le} and it is not possible to maximize H with respect to C_{Le}. Similarly for S_p equal to zero, H cannot be minimized with respect to C_{Lp}. If S_e or S_p are zero over a non-zero time interval, a singular control in C_{Le} or C_{Lp} may occur. Necessary conditions for the existence of singular solutions were presented in Chapter III. For a singular control in C_{Lp} it is necessary that S_p and all its time derivatives vanish. Similarly, for a singular control in C_{Le}, it is necessary that S_e and all time derivatives of S_e vanish. These conditions are now examined to determine if singular solutions exist for this problem.

Setting $S_p = 0$ yields

$$S_p = \lambda_{\gamma p} D_p / V_p = 0$$

which implies that $\lambda_{\gamma p} = 0$

Setting $\dot{S}_p = 0$ yields

$$\dot{S}_p = D_p [\dot{\lambda}_{\gamma p}/V_p - \lambda_{\gamma p} \dot{V}_p/V_p^2] = 0$$

which implies $D_p \dot{\lambda}_{\gamma p}/V_p = 0$ since $\lambda_{\gamma p} = 0$. Substituting for $\dot{\lambda}_{\gamma p}$,

$$D_p [\lambda_{xp} \sin \gamma_p - \lambda_{hp} \cos \gamma_p] = 0 \qquad (40)$$

which implies that $\tan \gamma_p = \lambda_{hp}/\lambda_{xp}$ along a singular arc

Setting $\ddot{S}_p = 0$ yields

$$\ddot{S}_p = D_p \dot{\gamma}_p [\lambda_{xp} \cos \gamma_p + \lambda_{hp} \sin \gamma_p]$$

$$+ D_p [\dot{\lambda}_{xp} \sin \gamma_p - \dot{\lambda}_{hp} \cos \gamma_p] = 0$$

$$D_p \dot{\gamma}_p [\lambda_{xp} \cos \gamma_p + \lambda_{hp} \sin \gamma_p] = 0$$

$$D_p^2 C_{Lp} [\lambda_{xp} \cos \gamma_p + \lambda_{hp} \sin \gamma_p]/V_p = 0$$

since $\dot{\lambda}_{xp}$ and $\dot{\lambda}_{hp}$ are zero from the costate differential equations. For $\ddot{S}_p$ to be zero, either C_{Lp} must be zero or

$$[\lambda_{xp} \cos \gamma_p + \lambda_{hp} \sin \gamma_p] = 0$$

Assuming $[\lambda_{xp} \cos \gamma_p + \lambda_{hp} \sin \gamma_p]$ equals zero implies that $\tan \gamma_p = - \lambda_{xp}/\lambda_{hp}$ along a singular arc. This is in contradiction with the result of Eq (40). It is concluded that along a singular arc the

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singular control C_{Lps} is equal to zero. A necessary condition for the optimality of this solution is that

$$-\partial \ddot{S}_p/\partial C_{Lp} \geq 0 \qquad (41)$$

Application of this condition gives

$$-\ddot{S}_p/\partial C_{Lp} = -[\lambda_{xp} \cos \gamma_p + \lambda_{hp} \sin \gamma_p] \, (D_p^2/V_p) \qquad (42)$$

To show when this necessary condition holds, transversality conditions require

$$\lambda_{xp} \, (t_f) = -(x_e - x_p)/R \,\Big|_{t=t_f}$$

and

$$\lambda_{hp} \, (t_f) = -(h_e - h_p)/R \,\Big|_{t=t_f}$$

Defining θ as the angle measured between a line from the pursuer to the evader and the horizontal at termination $\lambda_{xp} \, (t_f)$ and $\lambda_{hp} \, (t_f)$ become

$$\lambda_{xp} \, (t_f) = \lambda_{xp} \, (t) = - \cos \theta$$

$$\lambda_{hp} \, (t_f) = \lambda_{hp} \, (t) = - \sin \theta$$

since λ_{xp} and λ_{hp} are constants. Making these substitutions in Eq (42) gives

$$-\partial \ddot{S}_p/\partial C_{Lp} = \cos (\theta - \gamma_p) \qquad (43)$$

Along a singular arc $\dot{S}_p = 0$ requires

$$[\lambda_{xp} \sin \gamma_p - \lambda_{hp} \cos \gamma_p] = \sin (\theta - \gamma_p) = 0$$

which implies that $\gamma_p = \theta$ or $\gamma_p = \theta + \pi$. Substituting $\gamma_p = \theta$ and $\gamma_p = \theta + \pi$ into Eq (43) yields

$$-\partial \ddot{S}_p/\partial C_{Lp} = 1 > 0 \quad \text{for} \quad \gamma_p = \theta$$

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and

$$-\partial \ddot{S}_p / \partial C_{Lp} = -1 < 0 \quad \text{for} \quad \gamma_p = 0 + \pi$$

Therefore the necessary condition Eq (41) is only satisfied when $\gamma_p = 0$. A similar analysis for the evader reveals that $S_e = 0$ over a non-zero time interval corresponds to a singular solution with the singular control $C_{Les} = 0$.

The optimal controls for the solution to this differential game problem can now be given as

$$C_{Lp}^* = \begin{bmatrix} \overline{C}_{Lp} & ; & S_p \neq 0 \\ \\ 0 & ; & S_p = 0 \end{bmatrix}$$

$$C_{Le}^* = \begin{bmatrix} \overline{C}_{Le} & ; & S_e \neq 0 \\ \\ 0 & ; & S_e = 0 \end{bmatrix}$$

(44)

where $\overline{C}_{Lp}$ and $\overline{C}_{Le}$ are given by Eqs (39).

Problem Solution

In an optimal solution to this problem there are three choices of control depending on whether S_p and S_e are greater than, equal to, or less than zero. In order to construct an optimal solution to this problem the allowable sequences of these controls must be determined.

The first question to consider is whether or not a singular arc can be followed by a non-singular arc in an optimal sequence of controls. To answer this question assume that the pursuer is on a singular arc and applies the control $C_{Lp\ max}$. On the singular arc

the following is true.

(1) $\gamma_p = 0$

(2) $\lambda_{\gamma p} = 0$

(3) $\dot{\lambda}_{\gamma p} = V_p \sin(\theta - \gamma_p) = 0$

From Eqs (37) one sees that the application of the control $C_{Lp\ max}$ causes γ_p to increase so that $\dot{\gamma}_p > 0$. This implies that $\lambda_{\gamma p} < 0$ and that $\lambda_{\gamma p}$ also becomes negative. In order to satisfy the transversality condition $\lambda_{\gamma p}(t_f) = 0$ the sign of $\dot{\lambda}_{\gamma p}$ must somewhere become positive. This will happen when $\gamma_p = -\theta$. Physically this means that the flight direction of the pursuer, when $\dot{\lambda}_{\gamma p} = 0$, is in the opposite direction of the line of sight between the pursuer and evader at termination. At this point the transversality condition $\lambda_{\gamma p}(t_f) = 0$ still has not been satisfied since $\lambda_{\gamma p} < 0$. This implies continued application of the control $C_{Lp\ max}$ which causes further rotation of the flight path direction of the pursuer until $\lambda_{\gamma p} = 0$. If at this point $\dot{\lambda}_{\gamma p} = 0$, the pursuer will again be on a singular arc with $\gamma_p = 0$. This means that the pursuer has literally traveled a circular trajectory which obviously cannot be an optimal pursuit trajectory.

If $\dot{\lambda}_{\gamma p} > 0$ when $\lambda_{\gamma p} = 0$ the following control sequence must be considered

$$\{C_{Lp\ max}, C_{Lp\ min}\}$$

To show that this cannot be an optimal control sequence, assume that $\dot{\lambda}_{\gamma p} < 0$ and $\lambda_{\gamma p} > 0$. The choice of control corresponding to $\lambda_{\gamma p} < 0$ is $C_{Lp\ max}$. If $\dot{\lambda}_{\gamma p} > 0$ when $\lambda_{\gamma p} = 0$, $\lambda_{\gamma p}$ will become positive resulting in switching to $C_{Lp} = C_{Lp\ min}$. Since $\dot{\lambda}_{\gamma p} > 0$, γ_p must be

less than 0. The choice of $C_{Lp\ min}$ will cause γ_p to decrease and $\dot{\lambda}_{\gamma p}$ will remain positive causing $\lambda_{\gamma p}$ to become more positive. Again the pursuer will be traveling in a circle which physically cannot be an optimal pursuit trajectory. By symmetry, $\{C_{Lp\ min}, C_{Lp\ max}\}$ cannot be an optimal control sequence.

It is concluded that in an optimal sequence of controls a singular arc cannot be followed by a non-singular arc. Therefore if the pursuer is on a singular arc, he remains on the singular arc.

Having eliminated $\{C_{Lp\ max}, C_{Lp\ min}\}$, $\{C_{Lp\ min}, C_{Lp\ max}\}$, and the transition from a singular to a non-singular arc as optimal control sequences, there are three control sequences possible in an optimal solution to this problem. These are

$$\{C_{Lp\ max}\ or\ C_{Lp\ min},\ 0\}\ ;\ \{C_{Lp\ max}\ or\ C_{Lp\ min}\}\ ;\ \{0\}$$

A physical interpretation of the use of these control sequences can be made by an analysis similar to that used to show that a switch from a singular arc to a non-singular arc cannot be made. Consider an initial value of $\gamma_p < 0$. This implies that

$$\dot{\lambda}_{\gamma p} = V_p\ \sin\ (\theta - \gamma_p) > 0$$

To satisfy the transversality condition $\lambda_{\gamma p}\ (t_f) = 0$, $\lambda_{\gamma p}$ must be negative for which the choice of $C_{Lp} = C_{Lp\ max}$ is the optimal control. The control $C_{Lp\ max}$ rotates the flight path direction of the pursuer toward θ. When $\gamma_p = 0$, the pursuer switches to the singular control $C_{Lps} = 0$. For an initial value of $\gamma_p > 0$, the optimal control is $C_{Lp\ min}$ and the flight path direction of the pursuer is again rotated toward the direction θ. Similar arguments and statements can be made in regard to the optimal control sequences for the evader.

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The main point of this discussion is that the optimal solution to this problem involves the employment of maximum or minimum control to to rotate the velocity vector of each vehicle into a prescribed direction after which a singular $C_{Lp}^* = 0$ or $C_{Le}^* = 0$ is the optimal control. The trajectories of the pursuer and evader are characterized by a hard turn followed by a non-turning arc. The condition when both aircraft are on singular arcs occurs when the flight path directions of the two aircraft are colinear. This can be shown as follows.

On the singular arc, the state equations reduce to

$$\dot{x}_p = V_p \cos \theta \qquad\qquad \dot{x}_e = V_e \cos \theta$$

$$\dot{h}_p = V_p \sin \theta \qquad\qquad \dot{h}_e = V_e \sin \theta$$

$$\dot{V}_p = D_p C_{Tp} \qquad\qquad \dot{V}_e = D_e C_{Te}$$

$$\dot{\gamma}_p = 0 \qquad\qquad \dot{\gamma}_e = 0$$

Integrating $\dot{x}_p$ and $\dot{h}_p$ yields

$$x_p(t_f) = x_p(t) + \cos \theta \int_t^{t_f} V_p \, dt$$

$$h_p(t_f) = h_p(t) + \sin \theta \int_t^{t_f} V_p \, dt$$

and similarly for the evader

$$x_e(t_f) = x_e(t) + \cos \theta \int_t^{t_f} V_e \, dt$$

$$h_e(t_f) = h_e(t) + \sin \theta \int_t^{t_f} V_e \, dt$$

Dividing $[h_e(t_f) - h_p(t_f)]$ by $[x_e(t_f) - x_p(t_f)]$ yields

$$\tan \theta = \cfrac{h_e(t) + \sin\theta \int_t^{t_f} V_e \, dt - h_p(t) - \sin\theta \int_t^{t_f} V_p \, dt}{x_e(t) + \cos\theta \int_t^{t_f} V_e \, dt - x_p(t) - \cos\theta \int_t^{t_f} V_p \, dt}$$

which is equivalent to

$$\tan \theta = [h_e(t) - h_p(t)]/[x_e(t) - x_p(t)] \qquad (45)$$

This implies that the line of sight direction between the vehicles is a constant and equal to the terminal line of sight direction when both vehicles are on singular arcs. This proves that the flight path directions of both aircraft are colinear under these conditions.

Because of the special conditions associated with the singular arc solutions, initial conditions generally require the initial employment of maximum or minimum control. The closed-loop solution to a general problem can be constructed as follows:

(1) For the evader, choose the control $C_{Le \ max}$ or $C_{Le \ min}$ that causes the evader to rotate its velocity vector away from the pursuer. Holding this choice of control constant, integrate a turning trajectory for the evader.

(2) For the pursuer, choose the control $C_{Lp \ max}$ or $C_{Lp \ min}$ that causes the pursuer to turn in the direction necessary to achieve a tail chase condition. Holding this choice of control constant, integrate a turning trajectory for the pursuer.

(3) The points of tangency between the two trajectories identify the switching points for each aircraft. The slope of a line drawn between these two points identifies the final line of sight direction θ. The geometry of

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this solution is shown in Figure 7. If points of
tangency do not exist due to the ability of the evader
to turn tighter than the pursuer, the aircraft does not
switch from the turning controls to the singular
controls. The hard turn controls are then maintained
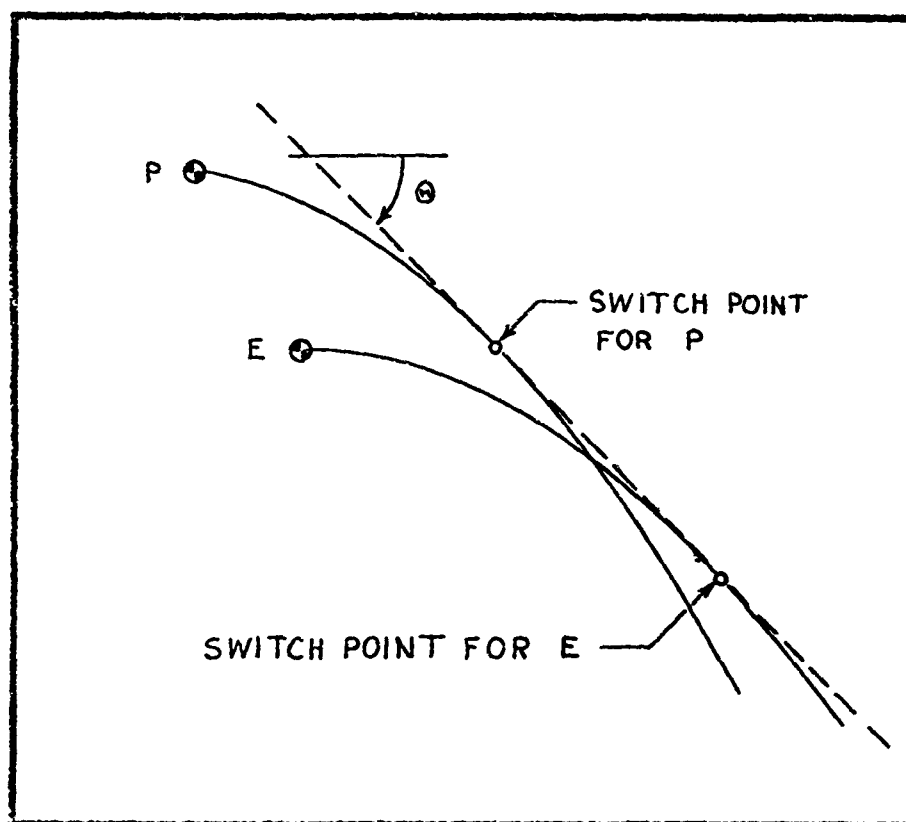over the complete trajectory.

Figure 7. Switching Point Geometry for Zero Induced Drag Control Law

This is an open-loop solution. However, it can be provided on an
instantaneous and continuous basis thereby producing a closed-loop
strategy. It is interesting to note that even though the problem

was presented as a fixed time problem, the same solution applies to the minimum time intercept problem.

This problem was formulated neglecting gravitational forces since this is sufficient to prevent the synthesis of closed-loop controls. However, it is felt that this problem and solution is of value since it can be argued that gravity effects both vehicles almost equally providing velocity differences are not too great. The most significant assumption related to this model is the neglecting of induced drag. Because of the distinctive two arc solution of this problem, a correction for induced drag can very easily be introduced. This correction consists of subtracting from the net thrust (T_e and T_p) an additional drag value corresponding to that normally induced by the value of $C_{Lp\ max}$ or $C_{Le\ max}$. This additional term would be required only during the turning portion of the solution.

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VII. PURSUIT-EVASION DIFFERENTIAL GAME - LINEARIZED DRAG POLAR MODEL

In the previous chapter the zero-induced drag model was introduced. It was pointed out how the drag due to lift or induced drag could be taken into account in that solution to the differential game problem. In this chapter, the aircraft model is an improvement over the zero-induced drag model in that drag due to lift is considered directly. The difference between this model and the standard aircraft model is the choice of a linearized drag polar instead of the parabolic drag polar. If gravitational forces are neglected, a closed-loop control law can be obtained for the pursuit-evasion differential game with the linearized drag polar model. The vectogram for this model is shown in Figure 8.

The purpose of this chapter is to solve the pursuit-evasion differential game with the linearized drag polar aircraft model.

Statement of the Problem

The problem is to determine a saddle point of

$$J(t_f) = R(t_f) = [(x_e - x_p)^2 + (h_e - h_p)^2]^{1/2}_{t=t_f}$$

subject to the state equations

$$\dot{x}_p = V_{xp}$$

$$\dot{h}_p = V_{hp}$$

$$\dot{V}_{xp} = D_p [(C_{Tp} - k_p | C_{Lp} |) \cos \gamma_p - C_{Lp} \sin \gamma_p] \tag{46}$$

$$\dot{V}_{hp} = D_p [(C_{Tp} - k_p | C_{Lp} |) \sin \gamma_p + C_{Lp} \cos \gamma_p]$$
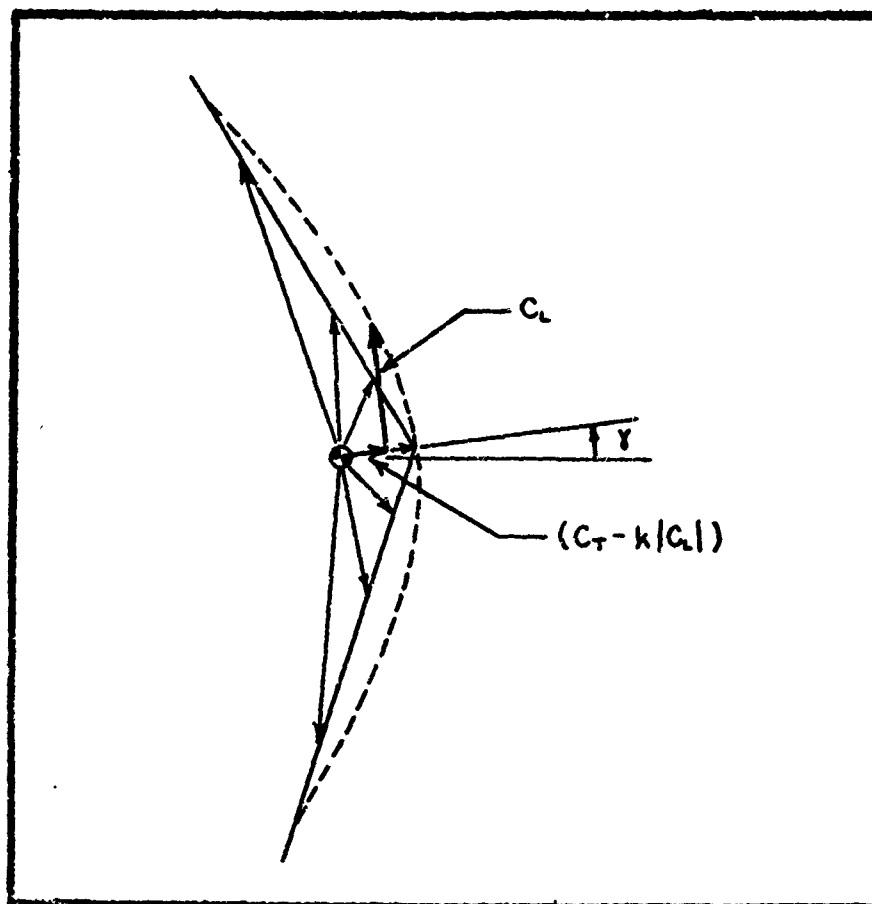
Figure 8. Acceleration Vectogram for the Linearized Drag Polar
Aircraft Model

$$\dot{x}_e = V_{xe}$$

$$\dot{h}_e = V_{he}$$

(47)

$$\dot{V}_{xe} = D_e \left[(C_{Te} - k_e \mid C_{Le} \mid) \cos \gamma_e - C_{Le} \sin \gamma_e \right]$$

$$\dot{V}_{he} = D_e \left[(C_{Te} - k_e \mid C_{Le} \mid) \sin \gamma_e + C_{Le} \cos \gamma_e \right]$$

where

$$\gamma_p \equiv \tan^{-1} (V_{hp}/V_{xp})$$

$$\gamma_e \equiv \tan^{-1} (V_{he}/V_{xe})$$

and the subscripts p and e refer to the pursuer and evader, respectively. The final time t_f as well as the initial values of the state variables are specified. $D_p C_{Tp}$, D_e and C_{Te} are constants defined in Chapter IV. C_{Le} and C_{Lp} are the controls and are constrained by

$$C_{Lp\ min} \leq C_{Lp} \leq C_{Lp\ max}$$

$$C_{Le\ min} \leq C_{Le} \leq C_{Le\ max}$$

The constants k_p and k_e are chosen so that the linearized drag polar relationships given by

$$C_{De} = C_{DOe} + k_e |C_{Le}|$$

$$C_{Dp} = C_{DOp} + k_p |C_{Lp}|$$

represent the best linear approximation to the parabolic drag polar relationships given in Chapter II. As in the last chapter, analysis of this problem is made easier if the state variables x, h, V_x, and V_h are replaced by V and γ where

$$V = [V_x^2 + V_h^2]^{1/2} \qquad \text{and} \qquad \gamma = \tan^{-1} (V_h/V_x)$$

The resulting state equations are

$$\dot{x}_p = V_p \cos \gamma_p$$

$$\dot{h}_p = V_p \sin \gamma_p \tag{48}$$

$$\dot{V}_p = D_p [C_{Tp} - k_p |C_{Lp}|]$$

$$\dot{\gamma}_p = D_p \, C_{Lp}/V_p$$

$$\dot{x}_e = V_e \cos \gamma_e$$

$$\dot{h}_e = V_e \sin \gamma_e$$

$$\dot{V}_e = D_e \, [C_{Te} - k_e \mid C_{Le} \mid]$$

$$\ddot{\gamma}_e = D_e \, C_{Le}/V_e$$

(49)

which replace Eqs (46) and (47).

Necessary Conditions

Applying necessary conditions for a saddle point solution, the Hamiltonian is

$$H = \lambda_{xe} \, V_e \cos \gamma_e + \lambda_{he} \, V_e \sin \gamma_e + \lambda_{Ve} \, D_e \, [C_{Te} - k_e \mid C_{Le} \mid]$$

$$+ \lambda_{\gamma e} \, D_e \, C_{Le}/V_e + \lambda_{xp} \, V_p \cos \gamma_p + \lambda_{hp} \, V_p \sin \gamma_p$$

$$+ \lambda_{Vp} \, D_p \, [C_{Tp} - k_p \mid C_{Lp} \mid] + \lambda_{\gamma p} \, D_p \, C_{Lp}/V_p$$

The costate equations are

$$\dot{\lambda}_{xp} = 0$$

$$\dot{\lambda}_{hp} = 0$$

(50)

$$\dot{\lambda}_{Vp} = - \lambda_{xp} \cos \gamma_p - \lambda_{hp} \sin \gamma_p + \lambda_{\gamma p} \, D_p \, C_{Lp}/V_p^2$$

$$\dot{\lambda}_{\gamma p} = V_p \, (\lambda_{xp} \sin \gamma_p - \lambda_{hp} \cos \gamma_p)$$

$$\dot{\lambda}_{xe} = 0$$

$$\dot{\lambda}_{he} = 0$$

$$\dot{\lambda}_{Vc} = -\lambda_{xe} \cos \gamma_e - \lambda_{he} \sin \gamma_e + \lambda_{\gamma e} D_e C_{Le}/V_e^2 \tag{51}$$

$$\dot{\lambda}_{\gamma e} = V_e (\lambda_{xe} \sin \gamma_e - \lambda_{he} \cos \gamma_e)$$

The transversality conditions yield

$$\lambda_{xe} (t_f) = -\lambda_{xp} (t_f) = (x_e - x_p)/R \ \big|_{t=t_f}$$

$$\lambda_{he} (t_f) = -\lambda_{hp} (t_f) = (h_e - h_p)/R \ \big|_{t=t_f}$$

$$\lambda_{Ve} (t_f) = 0 \tag{52}$$

$$\lambda_{\gamma e} (t_f) = 0$$

$$\lambda_{Vp} (t_f) = 0$$

$$\lambda_{\gamma p} (t_f) = 0$$

It is necessary that the Hamiltonian be minimized with respect to C_{Lp} and maximized with respect to C_{Le} subject to the constraints on C_{Lp} and C_{Le}. To determine the C_{Lp} that minimizes H, consider the terms in H that explicitly contain C_{Lp}. Defining these terms as $\overline{H}_p$,

$$\overline{H}_p = -\lambda_{Vp} D_p k_p \ | \ C_{Lp} \ | + \lambda_{\gamma p} D_p C_{Lp}/V_p \tag{53}$$

If $C_{Lp} > 0$, Eq (53) can be written as

$$\overline{H}_p^{+} = [-\lambda_{Vp} k_p D_p + \lambda_{\gamma p} D_p/V_p] C_{Lp}$$

and for $C_{Lp} < 0$, Eq (53) becomes

$$\overline{H}_p^{-} = [\lambda_{Vp} k_p D_p + \lambda_{\gamma p} D_p/V_p] C_{Lp}$$

Defining

$$M_p^+ \equiv [-\lambda_{Vp} k_p D_p + \lambda_{\gamma P} D_p/V_p] \qquad (54)$$

and

$$M_p^- \equiv [\lambda_{Vp} k_p D_p + \lambda_{\gamma p} D_p/V_p] \qquad (55)$$

Eq (53) can be written as

$$\bar{H}_p = \begin{cases} M_p^+ C_{Lp} & ; & C_{Lp} > 0 \\ \\ M_p^- C_{Lp} & ; & C_{Lp} < 0 \end{cases} \qquad (56)$$

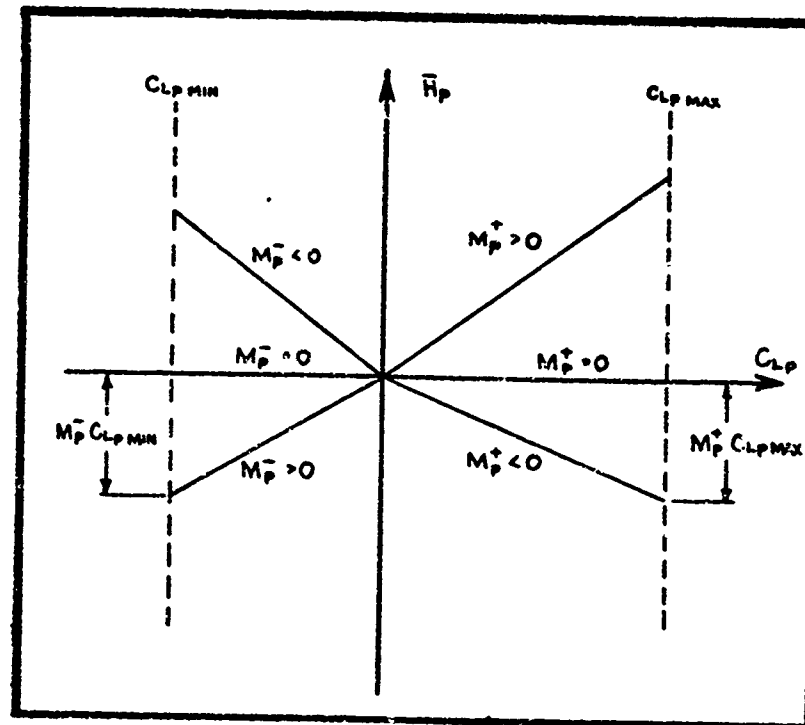To determine the C_{Lp} that minimizes $\bar{H}_p$ and thus minimizes H, the following figure is convenient.

Figure 9. Diagram Showing $\bar{H}_p$ vs. C_{Lp}

The minimum of $\overline{H}_p$ with respect to C_{Lp} is

$$\min \{0, M_p^- C_{Lp\ min}, M_p^+ C_{Lp\ max}\}$$

If the minimum of $\overline{H}_p$ is $M_p^- C_{Lp\ min}$, the minimizing control is $C_{Lp\ min}$.
If the minimum $\overline{H}_p$ is $M_p^+ C_{Lp\ max}$, the minimizing control is $C_{Lp\ max}$.
If the minimum $\overline{H}_p$ is equal to zero, the minimizing control C_{Lp} is either
zero or is a singular control.

To determine the C_{Le} that maximizes H define

$$\overline{H}_e = - \lambda_{Ve} D_e k_e |C_{Le}| + \lambda_{\gamma e} D_e C_{Le}/V_e$$

Defining

$$M_e^+ \equiv [- \lambda_{Ve} D_e k_e + \lambda_{\gamma e} D_e/V_e] \tag{57}$$

and

$$M_e^- \equiv [\lambda_{Ve} D_e k_e + \lambda_{\gamma e} D_e/V_e]. \tag{58}$$

$\overline{H}_e$ can be written as

$$\overline{H}_e = \begin{bmatrix} M_e^+ C_{Le} & C_{Le} > 0 \\ \\ M_e^- C_{Le} & C_{Le} < 0 \end{bmatrix} \tag{59}$$

The maximum $\overline{H}_e$ with respect to C_{Le} is

$$\max \{0, M_e^+ C_{Le\ max}, M_e^- C_{Le\ min}\}$$

If the maximum $\overline{H}_e$ is $M_e^+ C_{Le\ max}$, the maximizing control is $C_{Le\ max}$.
If the maximum $\overline{H}_e$ is $M_e^- C_{Le\ min}$, the maximizing control is $C_{Le\ min}$.
If the maximum $\overline{H}_e$ is zero, the maximizing control is either zero or
is a singular control.

Problem Solution

It is shown in Appendix B, that singular controls cannot occur in this problem and that an optimal solution is one of the control sequences

$$\{C_{Lp\ max}\ or\ C_{Lp\ min},\ 0\}\ ,\ \{C_{Lp\ max}\ or\ C_{Lp\ min}\}\ ,\ \{0\}$$

The switching point for the first sequence is determined by integrating backwards in time from the terminal surface the state and costate differential equations assuming the control $C_{Lp} = 0$

Along the $C_{Lp} = 0$ arc, the state equations are

$$\dot{x}_p = V_p \cos \gamma_p$$

$$\dot{h}_p = V_p \sin \gamma_p$$

$$\dot{V}_p = D_p C_{Tp} \tag{60}$$

$$\dot{\gamma}_p = 0$$

and the costate equations are

$$\dot{\lambda}_{xp} = 0$$

$$\dot{\lambda}_{hp} = 0$$

$$\dot{\lambda}_{Vp} = -\lambda_{xp} \cos \gamma_p - \lambda_{hp} \sin \gamma_p \tag{61}$$

$$\dot{\lambda}_{\gamma p} = V_p (\lambda_{xp} \sin \gamma_p - \lambda_{hp} \cos \gamma_p)$$

From transversality conditions and Eqs (61)

$$\lambda_{xp}(t) = \lambda_{xp}(t_f) = -\cos \theta$$

$$\lambda_{hp}(t) = \lambda_{hp}(t) = -\sin \theta \tag{62}$$

where θ is the angle measured between a line from the pursuer to the evader and the horizontal at $t = t_f$. Substituting Eqs (62) into (61) gives

$$\dot{\lambda}_{xp} = 0$$

$$\dot{\lambda}_{hp} = 0$$

$$\dot{\lambda}_{Vp} = \cos(\theta - \gamma_p) \qquad\qquad (63)$$

$$\dot{\lambda}_{\gamma p} = V_p \sin(\theta - \gamma_p)$$

Integrating $\dot{V}_p$ yields

$$V_p(t) = V_p(t_f) - D_p C_{Tp}(t_f - t) \qquad\qquad (64)$$

Integrating $\dot{\lambda}_{Vp}$ yields

$$\lambda_{Vp}(t) = -\cos(\theta - \gamma_p)(t_f - t) \qquad\qquad (65)$$

where $\lambda_{Vp}(t_f) = 0$ from transversality conditions.

Integrating $\dot{\lambda}_{\gamma p}$ yields

$$\lambda_{\gamma p}(t_f) - \lambda_{\gamma p}(t) = \sin(\theta - \gamma_p) \int_t^{t_f} V_p(t)\, dt$$

$$\lambda_{\gamma p}(t) = -\sin(\theta - \gamma_p) \int_{V_p(t)}^{V_p(t_f)} V_p(t)/(D_p C_{Tp})\, dV \qquad\qquad (66)$$

$$= -\sin(\theta - \gamma_p)\{V_p^2(t_f) - V_p^2(t)\}/(2D_p C_{Tp})$$

where $\lambda_{\gamma p}(t_f) = 0$ from transversality conditions. Substituting Eq (64) into (66),

$$\lambda_{\gamma p}(t) = -V_p(t)\sin(\theta - \gamma_p)(t_f - t)\{1 + \qquad\qquad (67)$$

$$D_p\ C_{Tp}\ (t_f - t)/(2V_p\ (t))\}$$

Necessary conditions to be on a $C_{Lp} = 0$ arc are

$$M^+ = \{- \lambda_{Vp}\ k_p + \lambda_{\gamma p}/V_p\} > 0 \tag{68}$$

or

$$M^- = \{\lambda_{Vp}\ k_p + \lambda_{\gamma p}/V_p\} < 0 \tag{69}$$

Substituting Eqs (65) and (67) into (68) and (69) yields

$$\tan (\theta - \gamma_p) < k_p/[1 + D_p\ C_{Tp}\ (t_f - t)/(2V_p)]$$

$$\tan (\theta - \gamma_p) > - k_p/[1 + D_p\ C_{Tp}\ (t_f - t)/(2V_p)]$$

which can be combined to give

$$|\ \tan (\theta - \gamma_p)\ | < k_p/[1 + D_p\ C_{Tp}\ (t_f - t)/(2V_p)] \tag{70}$$

The solution to a general problem is similar to the solution obtained in the last chapter in that it is characterized by a hard turn with either $C_{Lp\ max}$ or $C_{Lp\ min}$ as the control followed by a non-turning arc or dash. The principle difference between this solution and that of Chapter VI, is the condition under which switching occurs. In the solution to this problem switching to the non-turning arc occurs for the pursuer when

$$|\ \tan (\theta - \gamma_p)\ | = k_p/[1 + D_p\ C_{Tp}\ (t_f - t_{ps})/(2V_p)] \tag{71}$$

and for the evader when

$$|\ \tan (\theta - \gamma_e)\ | = k_e/[1 + D_e\ C_{Te}\ (t_f - t_{es})/(2V_?)] \tag{72}$$

where t_{ps} and t_{es} are the switching times for the pursuer and evader respectively, and θ must satisfy

$$\tan \theta = \{h_e(t_{es}) - h_p(t_{ps}) + V_{he}(t_{es})(t_f - t_{es})$$

$$- V_{hp}(t_{ps})(t_f - t_{ps}) + D_e C_{Te} \sin \gamma_e(t_{es})(t_f - t_{es})^2/_2$$

$$- D_p C_{Tp} \sin \gamma_p(t_{ps})(t_f - t_{ps})^2/_2\}/\{x_e(t_{es}) \tag{73}$$

$$- x_p(t_{ps}) + V_{xe}(t_{es})(t_f - t_{es}) - V_{xp}(t_{ps})(t_f - t_{ps})$$

$$+ D_e C_{Te} \cos \gamma_e(t_{es})(t_f - t_{es})^2/_2$$

$$- D_p C_{Tp} \cos \gamma_p(t_{ps})(t_f - t_{ps})^2/_2\}$$

To show this, the state equations given by Eqs (46) and (47) evaluated
for C_{Lp} and C_{Le} equal to zero are

$$\dot{x}_p = V_{xp} \qquad\qquad \dot{x}_e = V_{xe}$$

$$\dot{h}_p = V_{hp} \qquad\qquad \dot{h}_e = V_{he}$$

$$\dot{V}_{xp} = D_p C_{Tp} \cos \gamma_p \qquad\qquad \dot{V}_{xe} = D_e C_{Te} \cos \gamma_e$$

$$\dot{V}_{hp} = D_p C_{Tp} \sin \gamma_p \qquad\qquad \dot{V}_{he} = D_e C_{Te} \sin \gamma_e$$

Integrating $\dot{V}_{xp}$ and $\dot{V}_{xe}$ and remembering that γ_p and γ_e are constant
when C_{Lp} and C_{Le} are equal to zero yields

$$V_{xp}(t) = V_{xp}(t_{ps}) + D_p C_{Tp} \cos \gamma_p(t_{ps})(t - t_{ps})$$

and

$$V_{xe}(t) = V_{xe}(t_{es}) + D_p C_{Te} \cos \gamma_e(t_{es})(t - t_{es})$$

Integrating these equations and combining the results,

$$x_e(t) - x_p(t) = x_e(t_{es}) - x_p(t_{ps}) + V_{xe}(t_{es})(t - t_{es}) - \tag{74}$$

$$V_{xp} (t_{ps}) (t - t_{ps}) + D_e C_{Te} \cos \gamma_e (t_{es}) (t - t_{es})^2/_2$$

$$- D_p C_{Tp} \cos \gamma_p (t_{ps}) (t - t_{ps})^2/_2$$

The following equation is similarly obtained for the h components of the state equations.

$$h_e (t) - h_p (t) = h_e (t_{es}) - h_p (t_{ps}) = V_{he} (t_{es}) (t - t_{es}) \tag{75}$$

$$- V_{hp} (t_{ps}) (t - t_{ps}) + D_e C_{Te} \sin \gamma_e (t_{es}) (t - t_{es})^2/_2$$

$$- D_p C_{Tp} \sin \gamma_p (t_{ps}) (t - t_{ps})^2/_2$$

Dividing Eq (75) by (74) and evaluating at $t = t_f$ yields

$$\tan G = \{ h_e (t_{es}) - h_p (t_{ps}) + V_{he} (t_{es}) (t_f - t_{es})$$

$$- V_{hp} (t_{ps}) (t_f - t_{ps}) + D_e C_{Te} \sin \gamma_e (t_{es}) (t_f - t_{es})^2/_2$$

$$- D_p C_{Tp} \sin \gamma_p (t_{ps}) (t_f - t_{ps})^2/_2 \}/\{ x_e (t_{es})$$

$$- x_p (t_{ps}) + V_{xe} (t_{es}) (t_f - t_{es}) - V_{xp} (t_{ps}) (t_f - t_{ps})$$

$$+ D_e C_{Te} \cos \gamma_e (t_{es}) (t_f - t_{es})^2/_2$$

$$- D_p C_{Tp} \cos \gamma_p (t_{ps}) (t_f - t_{ps})^2/_2 \}$$

Because the optimal controls for this problem solution correspond to either a hard turn or a straight dash and because the switching point can be defined in terms of time to go and the state variables, a closed-loop solution to a general problem can be constructed as follows:

(1) For the evader, choose the control $C_{Le\ max}$ or $C_{Le\ min}$

that causes the evader to rotate its velocity vector away from the pursuer. Holding this choice of control constant, integrate a turning trajectory for the evader.

(2) For the pursuer, choose the control $C_{Lp\ max}$ or $C_{Lp\ min}$ that causes the pursuer to turn in the direction necessary to achieve a tail chase condition. Holding this choice of control constant integrate a turning trajectory for the pursuer.

(3) The switching points (if they exist) for the pursuer and evader must lie on these turning trajectories and must satisfy Eqs (71) and (72). A numerical search is required to find the direction θ and the pair of points on the two turning trajectories that satisfy Eqs (71), (72), and (73). If these switching points do not exist due to the ability of the evader to turn tighter than the pursuer, the aircraft will not switch from the turning controls to the zero controls and the hard turn controls will be maintained over the complete trajectory. The geometry of this solution is shown in Figure 10.

As in the previous chapter, this solution is an open-loop solution. However, since it can be provided on a continuous basis, it constitutes a closed-loop strategy or control law.

The importance of this differential game problem solution is that drag due to lift (induced drag) was included in the aircraft model.
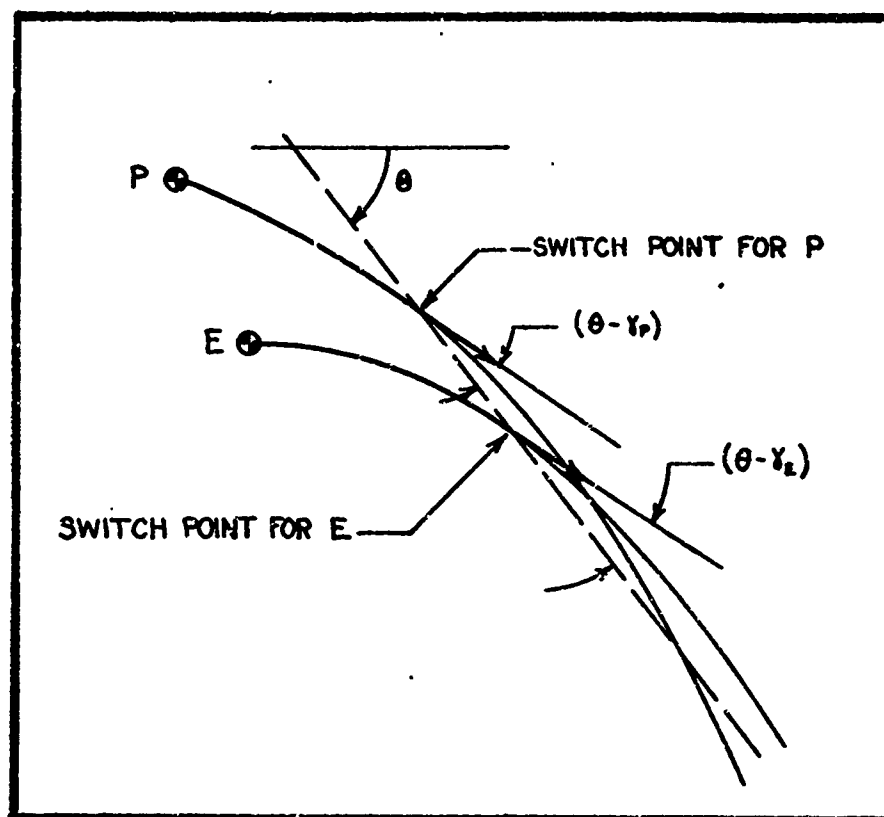
Figure 10. Switching Point Geometry for Linearized Drag Polar
Control Law

VIII. MODEL COMPARISON

The preceding four chapters were concerned with the solution to a pursuit-evasion differential game between two aircraft. The difference in these chapters was in the model chosen to represent the aircraft. The purpose of this chapter is to apply the results obtained in Chapters IV thru VII to the same game situation and to compare the different solutions. In doing this there are two important considerations. The first is the payoff or value of the game which is the numerical measure that reflects the relative capabilities of the two aircraft. The capability to determine this numerical measure through the use of simplified aircraft models would be a significant design tool. The second consideration is the correspondence of the resulting aircraft trajectories. The capability to determine optimal or near optimal trajectories by using closed-loop control laws developed through the use of simplified models is important in the area of aircraft tactics. To accomplish this comparison, two problems are considered. These are the problems for which open-loop solutions were presented in Chapter IV. In Chapter IV, these problems were defined by specifying final values of the state vector. It was pointed out that the solutions to these problems represented open-loop solutions for the differential game problems where the specified initial state conditions $\underline{x}(t_o)$ corresponded to some point $\underline{x}(t)$ on these trajectori. For purposes of this chapter, these problems are specified in terms of initial values of the state vector.

Problem 1

The initial conditions for this problem are

$$x_e(t_o) = 19522 \text{ feet} \qquad x_p(t_o) = 10893 \text{ feet}$$

$$h_e(t_o) = 27671 \text{ feet} \qquad h_p(t_o) = 39984 \text{ feet}$$

$$V_{xe}(t_o) = 857 \text{ ft/sec} \qquad V_{xp}(t_o) = 1137 \text{ ft/sec}$$

$$V_{he}(t_o) = 553 \text{ ft/sec} \qquad V_{hp}(t_o) = 508 \text{ ft/sec}$$

The choice of other necessary data is

$$D_e = 122 \text{ ft/sec}^2 \qquad D_p = 192 \text{ ft/sec}^2$$

$$C_{Te} = .06 \qquad C_{Tp} = .04$$

$$k_e = .5 \qquad k_p = .5$$

The final time is 20 seconds and the admissible controls are given by

$$|C_{Le}| \leq 1 \qquad |C_{Lp}| \leq 1$$

The standard model open-loop solution to this problem was shown in Figure 4. In this problem the pursuer is initially above and behind the evader. The pursuer has approximately a 200 ft/sec speed advantage over the evader and also has a greater acceleration capability by virtue of the choice of D_p and D_e.

Static Model Solution

In Chapter V, a procedure was presented that would enable the static model solution to be used as a closed-loop control law. The procedure is to solve the static problem at each integration step as if the flight path direction of both aircraft is to remain fixed. The optimal controls for the solution to the static problem are then applied to the standard model dynamics. Applying this control law to Problem 1

yields the solution shown in Figure 11. For comparative purposes, the standard model solution is also shown. The payoff or value of the game is 10055 feet.

Zero Induced Drag Model Solution

In Chapter VI, a zero induced drag model was presented. By assuming zero induced drag and neglecting gravitational forces, an optimal closed-loop control law was obtained. A method for correcting for induced drag was also discussed. For purposes of this chapter, gravity and the induced drag correction are included. Applying the control law obtained for the zero induced drag model to Problem 1, yields the solution in Figure 12. The value of the game is 10932 feet.

Linearized Drag Polar Model Solution

In Chapter VII, an optimal closed-loop solution was obtained for a linearized drag polar model with gravity neglected. For purposes of this chapter, gravity is included. Application of the linearized drag polar model control law to Problem 1, yields the solution shown in Figure 13. The value of the game is 11110 feet.

Problem 2

The initial conditions for this problem are

$$x_e(t_o) = 15036 \text{ feet} \qquad x_p(t_o) = 9944 \text{ feet}$$

$$h_e(t_o) = 23131 \text{ feet} \qquad h_p(t_o) = 19298 \text{ feet}$$

$$V_{xe}(t_o) = 1246 \text{ ft/sec} \qquad V_{xp}(t_o) = 1403 \text{ ft/sec}$$

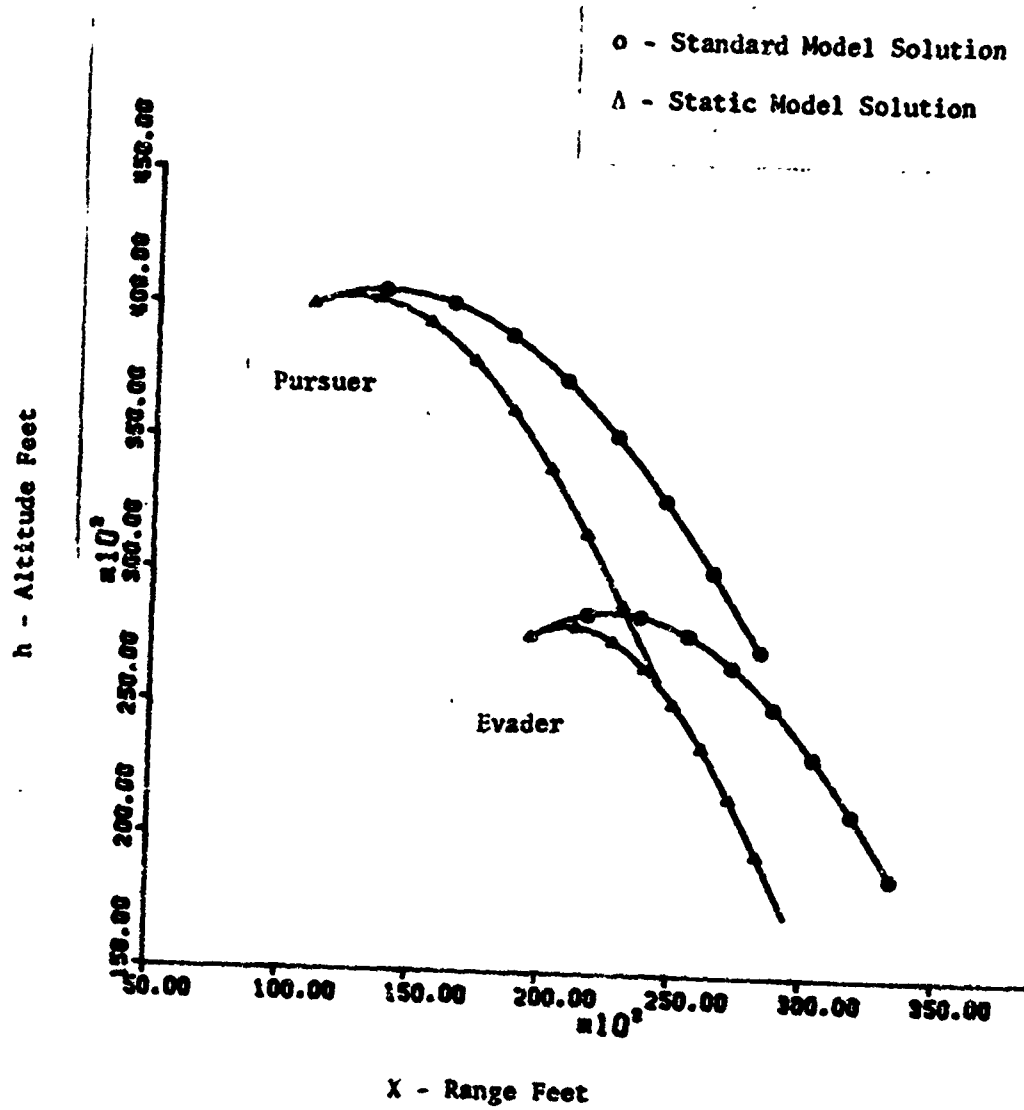$$V_{he}(t_o) = -72 \text{ ft/sec} \qquad V_{hp}(t_o) = -361 \text{ ft/sec}$$

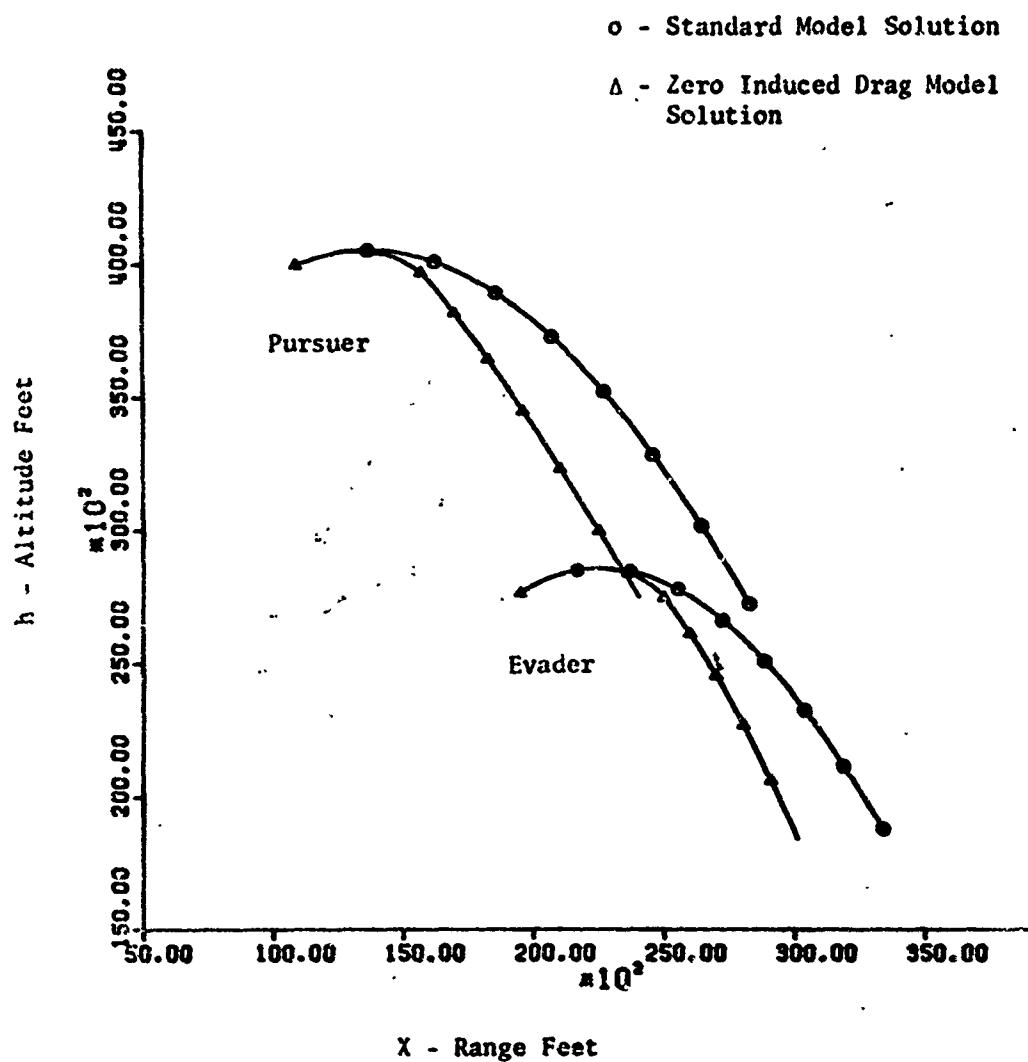Figure 11.   Static Model Solution - Problem 1
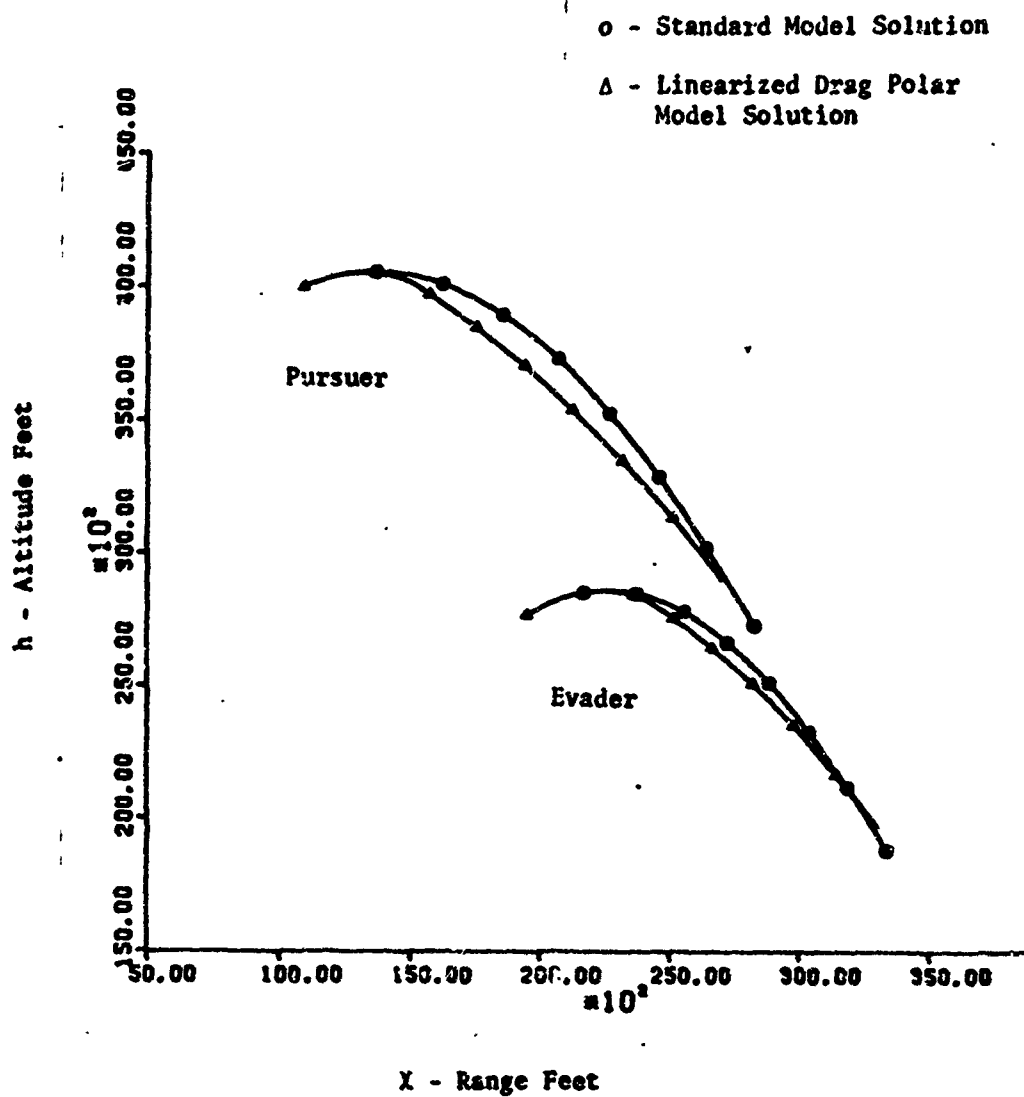
Figure 12. Zero Induced Drag Model Solution - Problem 1

Figure 13. Linearized Drag Polar Model Solution - Problem 1

The choice of other necessary data is

$$D_e = 160 \text{ ft/sec}^2 \qquad\qquad D_p = 175 \text{ ft/sec}^2$$

$$C_{Te} = .06 \qquad\qquad C_{Tp} = .04$$

$$k_e = .5 \qquad\qquad k_p = .5$$

The final time is 20 seconds and the admissible controls are given by

$$|C_{Le}| \leq 1 \qquad\qquad |C_{Lp}| \leq 1$$

The pursuing aircraft is initially below and behind the evading aircraft. Although the pursuer has an initial velocity advantage of approximately 200 ft/sec., his acceleration advantage over the evader is very small. The standard model open-loop solution to this problem was shown in Figure 5.

Applying the static model control law to this problem yields the solution shown in Figure 14. The value of the game is 5863 feet.

The zero induced drag model solution without the induced drag correction is shown in Figure 15. In this solution the "tail chase" situation is never achieved. Although the pursuer has a lift advantage over the evader, the turn rate of the pursuer is limited by his velocity, V_p, since turn rate is inversely proportional to velocity. When the induced drag correction is included, the pursuer's velocity is maintained sufficiently low to allow achievement of the tail chase solution. This solution is shown in Figure 16. The value of the game for these solutions is 5440 feet with the induced drag correction and 4302 feet without.

Three linearized drag polar model solutions are shown. The difference between these solutions is in the choice of the induced drag
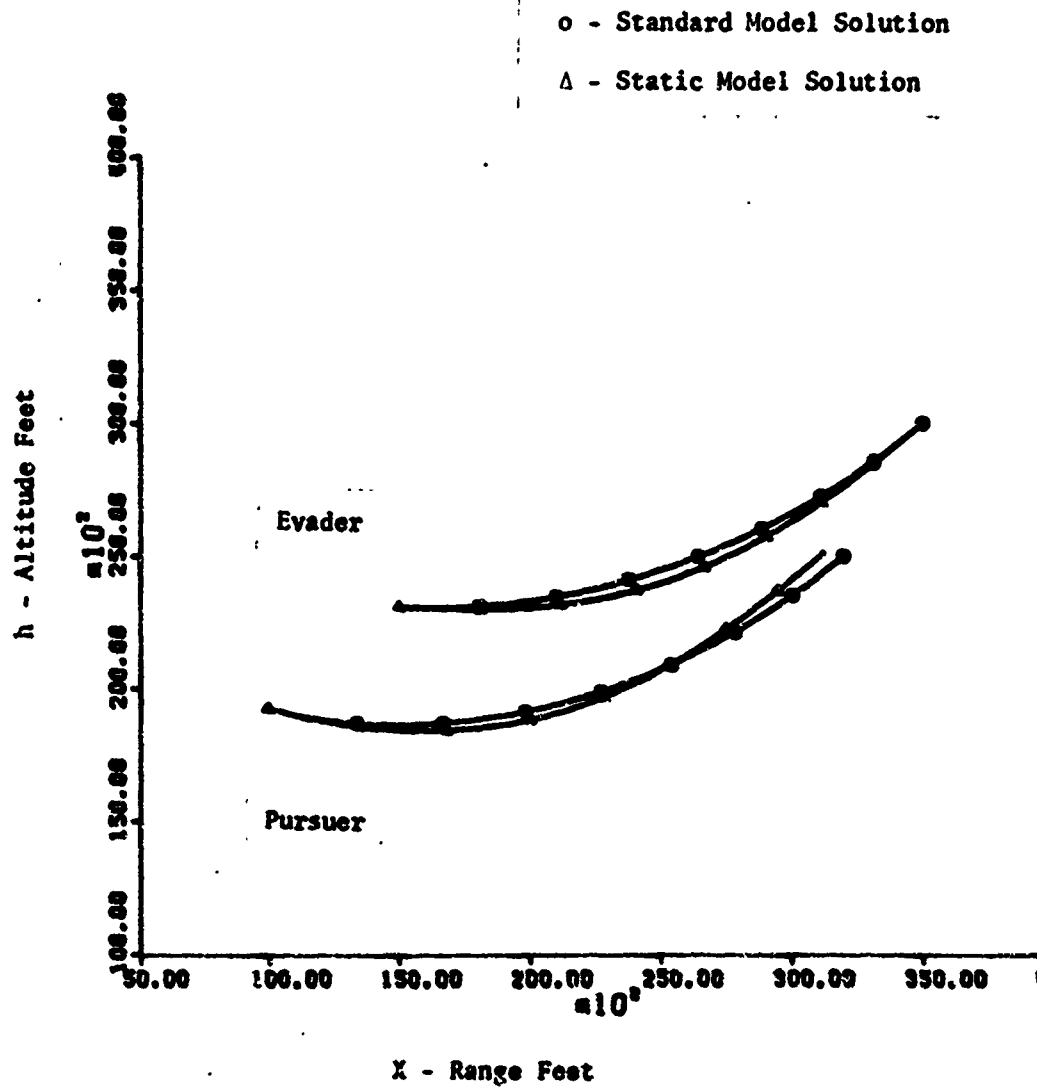
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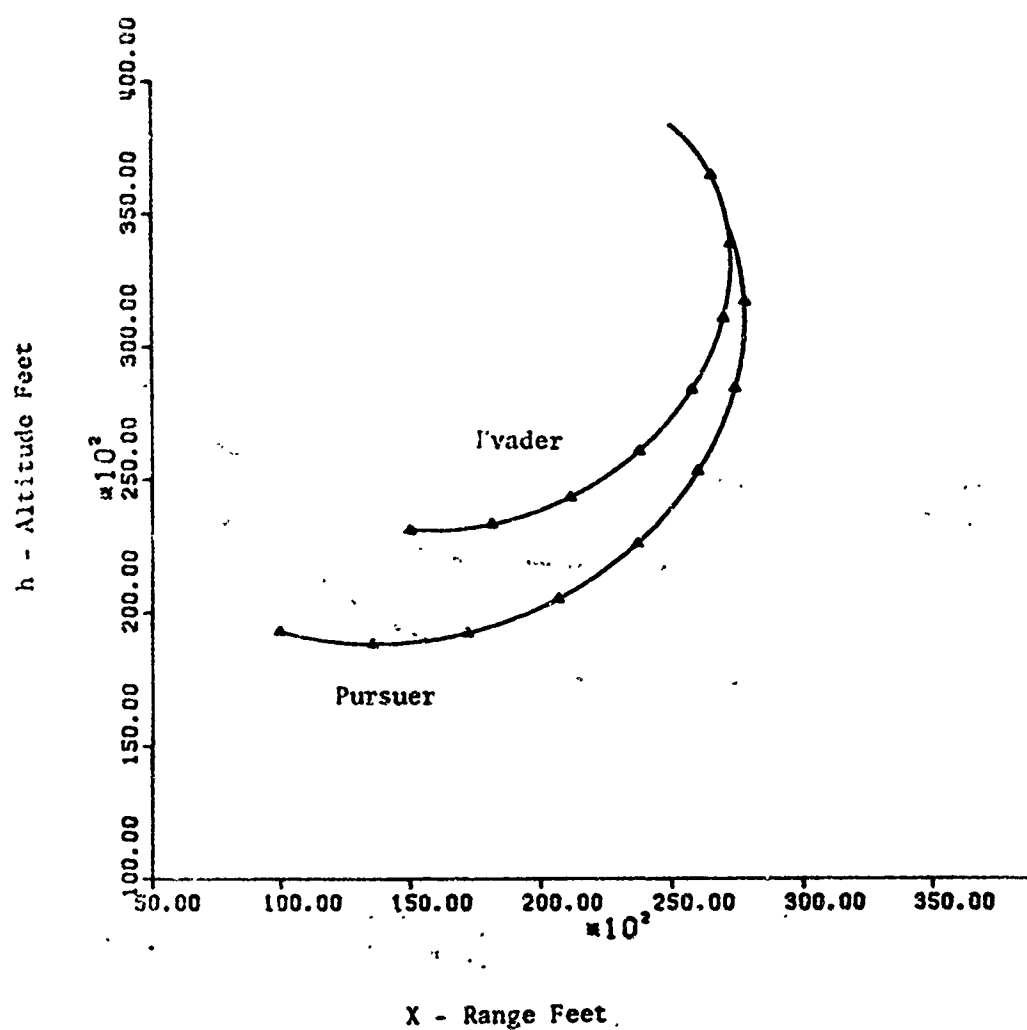Figure 14. Static Model Solution - Problem 2

Figure 15. Zero Induced Drag Model Solution without the Induced
Drag Correction - Problem 2
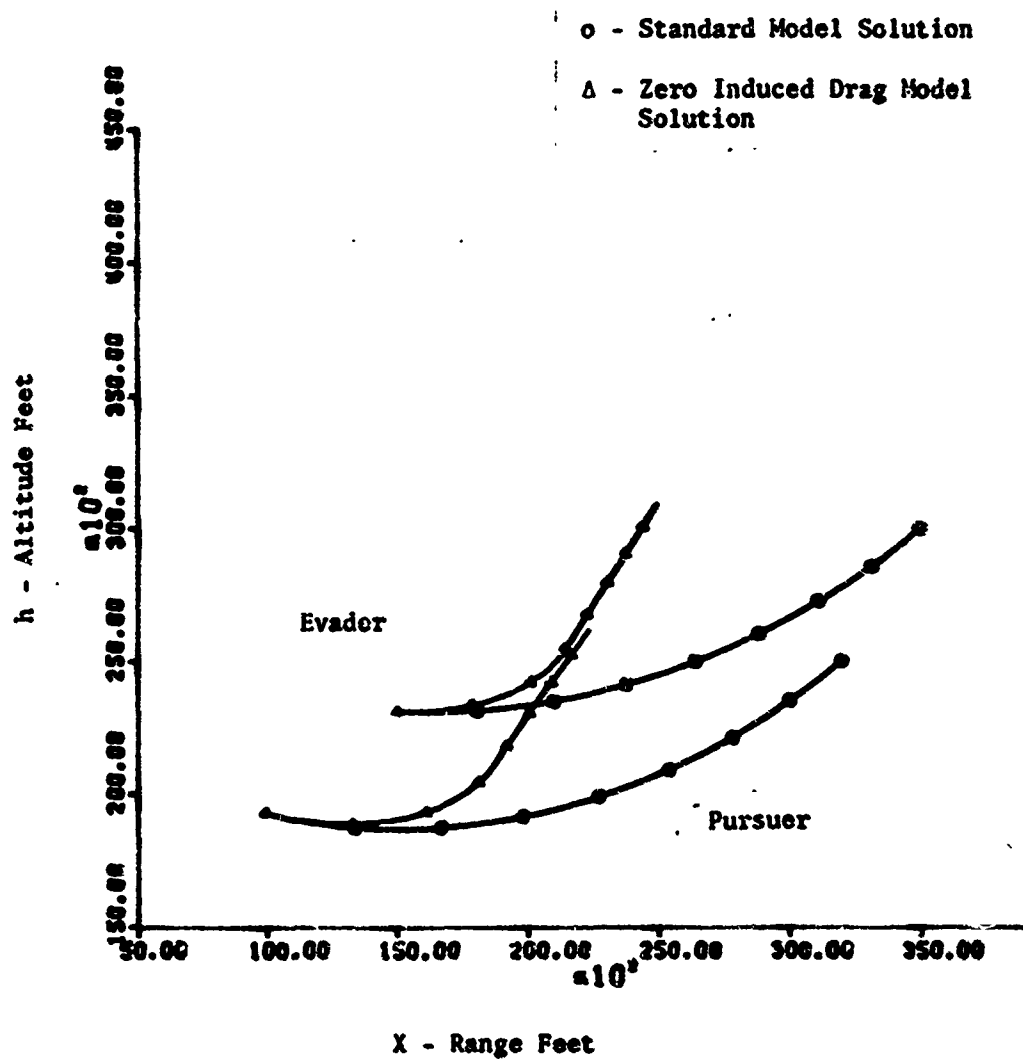
Figure 16. Zero Induced Drag Model Solution with the Induced
Drag Correction - Problem 2

coefficient (k). For the solution shown in Figure 17, the value of k_e and k_p is the same as the parabolic drag polar coefficient used in the standard model ($k_p = k_e = .5$). For the solution shown in Figure 18, k_p and k_e are taken to be .4 and in Figure 19, k_p and k_e are .33. The lower coefficients correspond to a lesser value of induced drag. The variations in these solutions give an indication of the importance of how induced drag is accounted for in the aircraft model. The value of the game for these solutions is

5798 feet for $k_p = k_e = .5$

5412 feet for $k_p = k_e = .4$

5140 feet for $k_p = k_e = .33$

Discussion

For convenience in comparing the results obtained in this chapter, the payoffs or game values for both problems are summarized. They are

Problem 1

Model	Payoff
Standard	9897 feet
Static	10055 feet
Zero Induced Drag	10932 feet
Linearized Drag Polar	11110 feet

Problem 2

Model	Payoff
Standard	5831 feet
Static	5863 feet

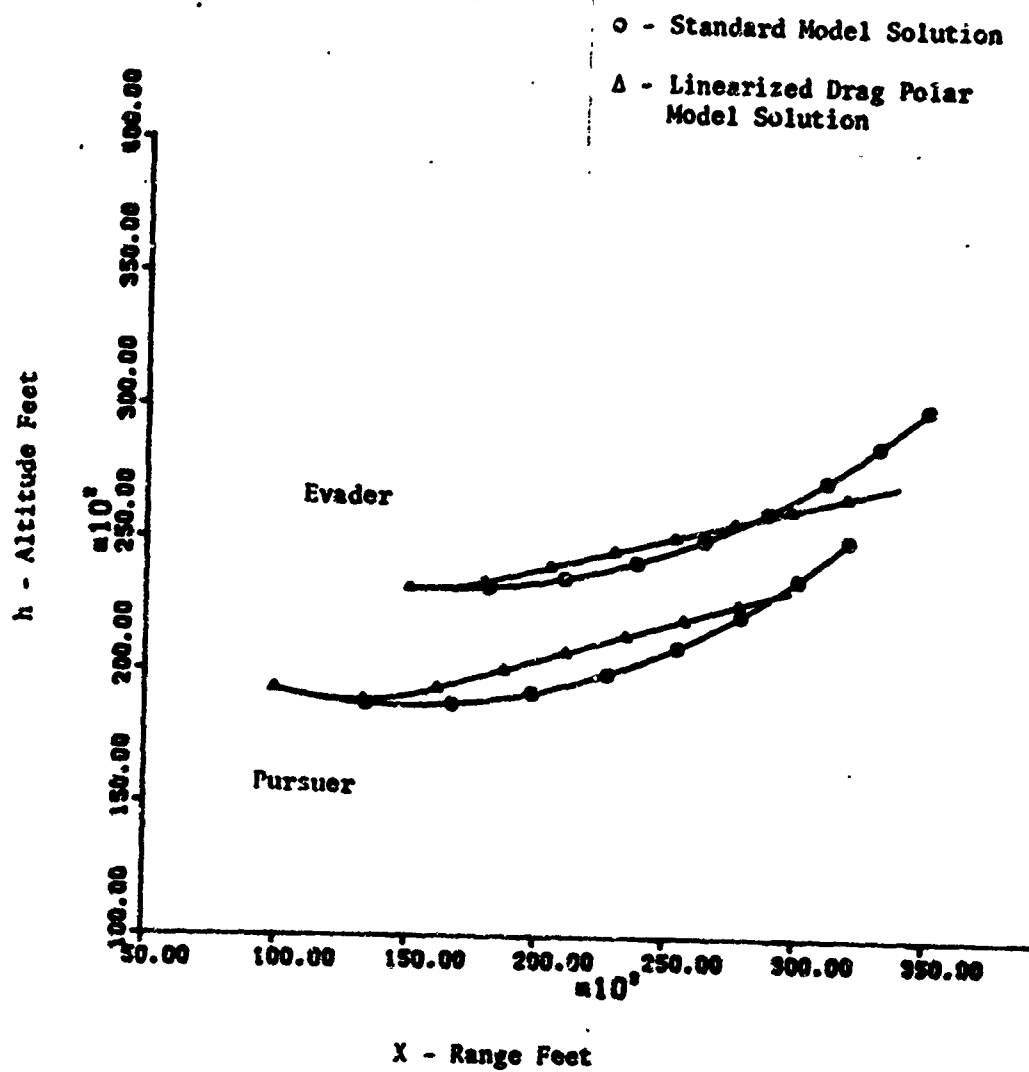Figure 17. Linearized Drag Polar Model Solution - Problem 2
$(k_p = k_e = .5)$
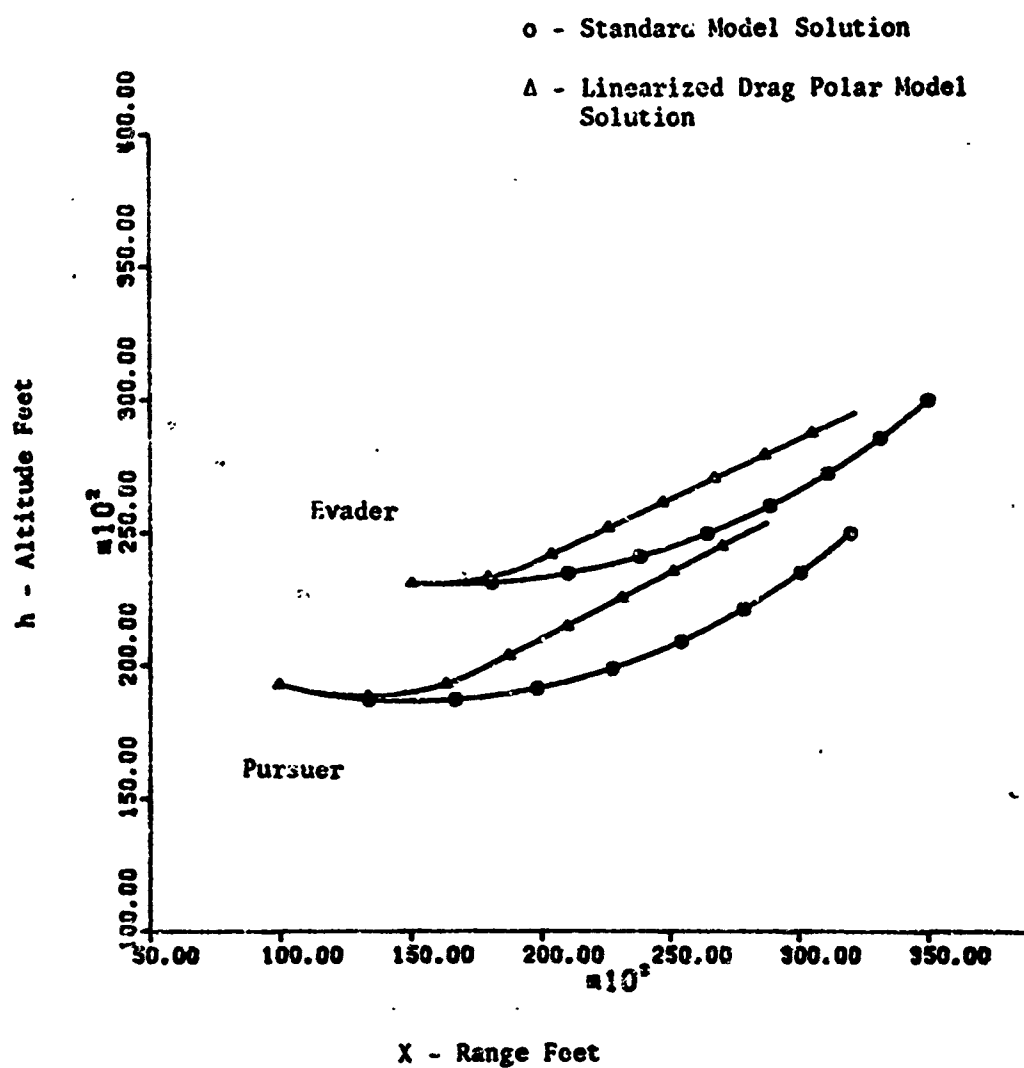
Figure 18.   Linearized Drag Polar Model Solution - Problem 2
$$(k_p = k_e = .4)$$
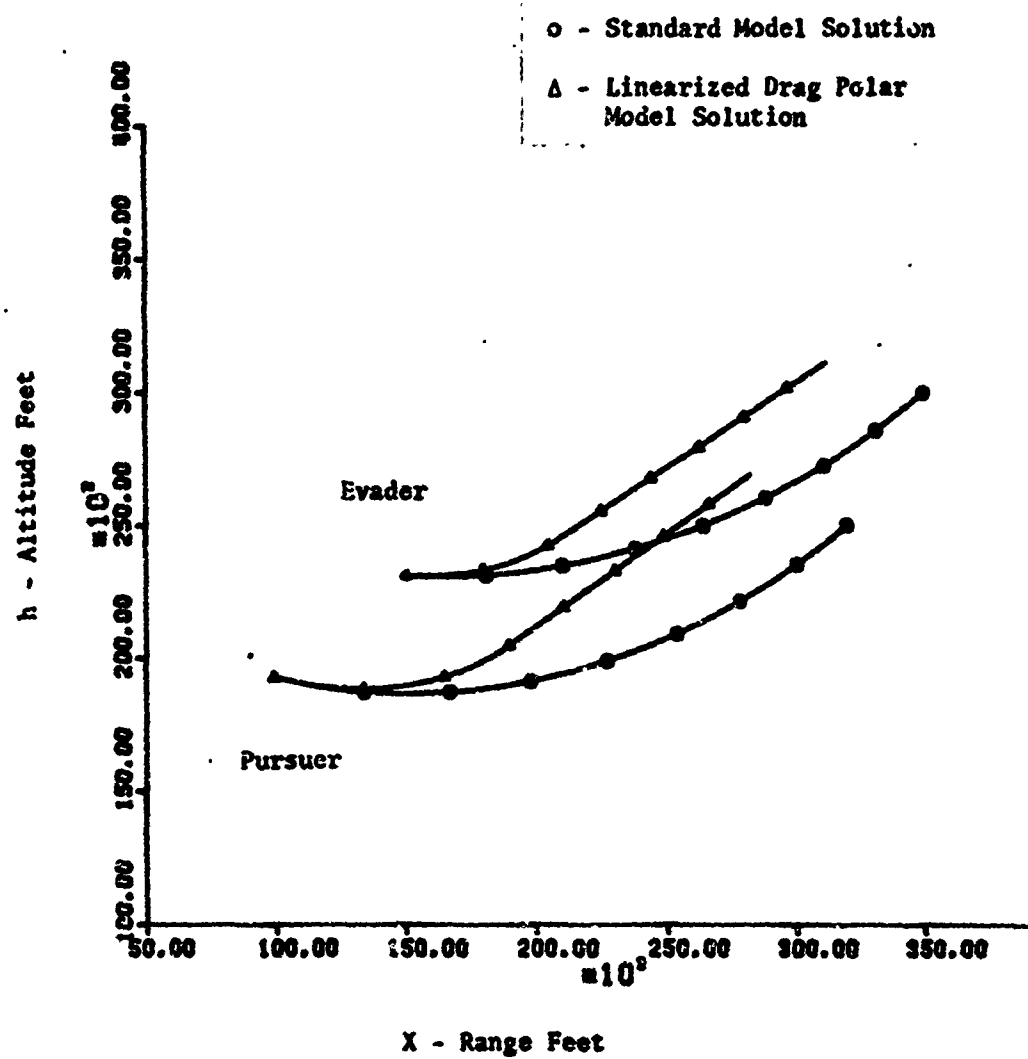
Figure 19. Linearized Drag Polar Model Solution - Problem 2
$(k_p = k_e = .33)$

Zero Induced Drag

    with correction   5440 feet

    without correction   4302 feet

Linearized Drag Polar

    $k_p = k_e = .5$ 5798 feet

    $k_p = k_e = .4$ 5412 feet

    $k_p = k_e = .33$ 5140 feet

These results indicate that a numerical measure reflecting the relative capability between two aircraft can be determined by solving differential games with simplified models. All of the solutions to Problem 1 result in trajectories that exhibit a distinct turning arc followed by a fairly straight approach to termination. The standard aircraft model solution to Problem 2 does not exhibit this characteristic as distinctly because of the effect of gravity. It is interesting to note that the line of sight angle (θ) between the pursuer and evader at termination is nominally independent of the model that is used.

These observations are used in the next chapter to synthesize a feedback control law for the non-linear standard aircraft model.

IX. CONTROL LAW SYNTHESIS

In Chapters V, VI, and VII, optimal closed-loop solutions to
special types of pursuit-evasion differential game problems were
presented. In order to obtain these solutions various simplifications
were made to the standard aircraft model. In Chapter VIII, the
simplified model solutions were compared with standard model solutions.
The purpose of this chapter is to synthesize a closed-loop control law
for the standard aircraft model utilizing information from the results
of Chapters V thru VIII.

Control Law Synthesis

The most important parameter in the solutions of the simplified
aircraft problems is θ, the line of sight angle between the two
aircraft at termination. Once θ is determined, the solution to these
problems is known. In the static model problem the optimal controls
are

$$C_{Lp}^{*} = \tan (\theta - \xi)/(2k_p)$$

and

$$C_{Le}^{*} = \tan (\theta - \psi)/(2k_e)$$

for

$$C_{Lp\ min} \leq C_{Lp}^{*} \leq C_{Lp\ max}$$

$$C_{Le\ min} \leq C_{Le}^{*} \leq C_{Le\ max}$$

Geometric significance can be given to this choice of controls
since the tangent to the acceleration vectogram of the static model at
the point corresponding to C_{Lp}^{*} is perpendicular to the final line of

sight direction θ. To show this the x and h coordinates corresponding
to this point on the vectogram are

$$x = D_p \left[(C_{Tp} - k_p C_{Lp}^{*2}) \cos \xi - C_{Lp}^* \sin \xi \right] \tag{76}$$

$$h = D_p \left[(C_{Tp} - k_p C_{Lp}^{*2}) \sin \xi + C_{Lp}^* \cos \xi \right] \tag{77}$$

Differentiating Eqs (76) and (77) with respect to C_{Lp}^*,

$$dx/dC_{Lp}^* = D_p \left[-2k_p C_{Lp}^* \cos \xi - \sin \xi \right] \tag{78}$$

and

$$dh/dC_{Lp}^* = D_p \left[-2k_p C_{Lp}^* \sin \xi + \cos \xi \right] \tag{79}$$

Combining Eqs (78) and (79),

$$dh/dx = \left[-2k_p C_{Lp}^* \sin \xi + \cos \xi \right]/\left[-2k_p C_{Lp}^* \cos \xi - \sin \xi \right]$$
$$= \left[-2k_p C_{Lp}^* \tan \xi + 1 \right]/\left[-2k_p C_{Lp}^* - \tan \xi \right] \tag{80}$$

with $C_{Lp}^* = \tan (\theta - \xi)/(2k_p)$ Eq (80) becomes

$$dh/dx = \left[-\tan (\theta - \xi) \tan \xi + 1 \right]/\left[-\tan (\theta - \xi) - \tan \xi \right]$$
$$= -1/\tan \theta$$

This proves that the tangent to the vectogram at the point corresponding to C_{Lp}^* is perpendicular to θ. Figure 20 shows that if a value of C_{Lp} greater or less than C_{Lp}^* is chosen, a lower component of acceleration is applied along the θ direction. This geometrically represents the pursuer's half of the saddle point solution. The optimal control for this problem can be defined as that admissible control which applies the largest acceleration component in the θ direction.

In the zero induced drag model problem the optimal solution is to employ a saturated control until the flight path direction of the
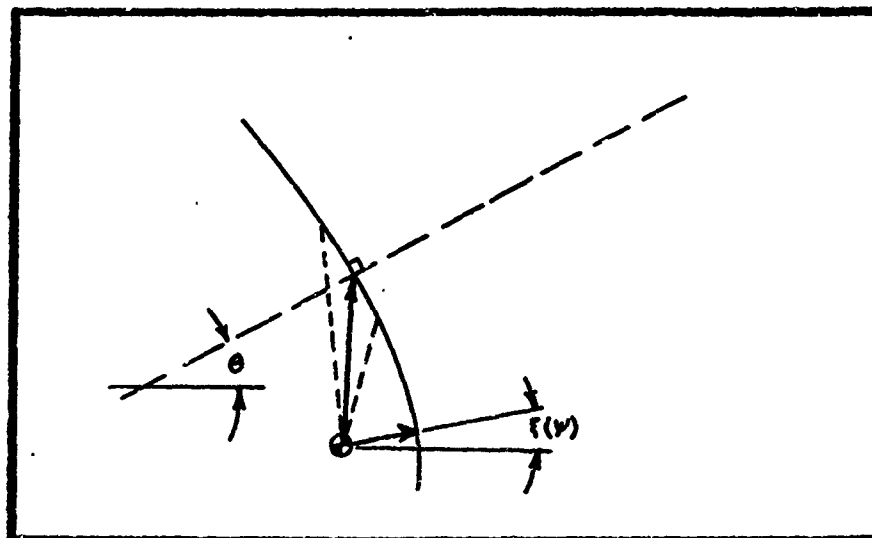
Figure 20. Static Model Control Selection Geometry

vehicles are equal to 0. The choice of the saturated control $(C_{L\ max})$ applies the largest acceleration component possible in the 0 direction as can be seen in Figure 21. When the flight path directions of the vehicles are in the 0 direction all points on the acceleration vectogram are perpendicular to the 0 direction. At this time any choice of control other than the singular control $C_{Lps} = 0$ will cause the aircraft to turn away from the desired direction of flight.

In the linearized drag polar model problem the optimal solution involves the employment of saturated controls until

$$| \tan (0 - \gamma_p)| < k_p /[1 + D_p \, C_{Tp} \, (t_f - t)/(2V_p)]$$

is satisfied for the pursuer and

$$| \tan (0 - \gamma_e)| < k_e /[1 + D_e \, C_{Te} \, (t_f - t)/(2V_e)]$$

is satisfied for the evader.
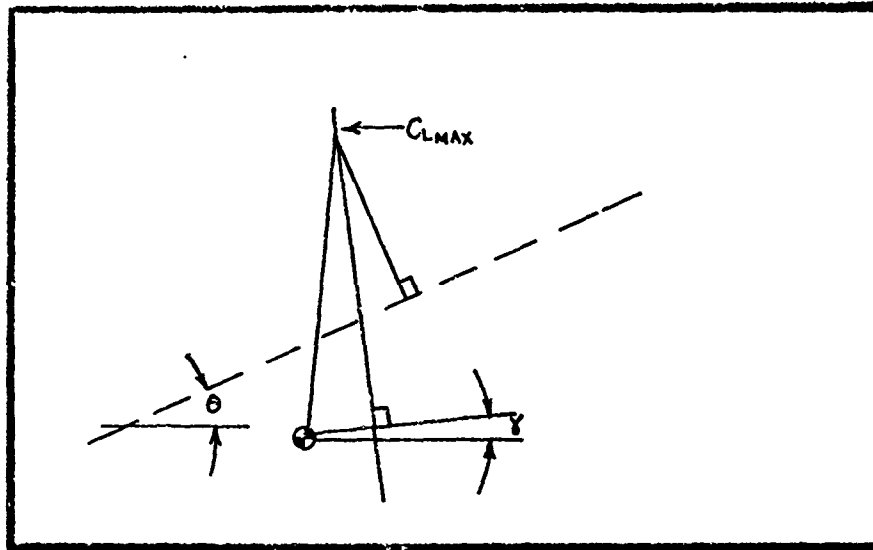
Figure 21. Zero Induced Drag Model Control Selection Geometry

When these inequalities are satisfied any gains made by continued application of the saturated control are more than offset by increased drag penalties.

In the standard aircraft model problem the optimal controls are

$$C_{Lp}^* = \tan (\beta_p - \gamma_p)/(2k_p)$$

and

$$C_{Le}^* = \tan (\beta_e - \gamma_e)/(2k_e)$$

for

$$C_{Lp \; min} \leq C_{Lp}^* \leq C_{Lp \; max}$$

$$C_{Le \; min} \leq C_{Le}^* \leq C_{Le \; max}$$

The difference between this problem and the static problem is that β_p, γ_p, β_e, and γ_e are not constants. The analysis of the optimal controls for the static model does apply on an instantaneous basis since $\overset{*}{C}_{Lp}$ and $\overset{*}{C}_{Le}$ are the controls that apply the largest acceleration component along the β_p and β_e directions respectively. This suggests a method of synthesizing a closed-loop control law for the standard aircraft model by approximating instantaneous values of β_p and β_e. From transversality conditions for the standard model problem

$$\beta_p \ (t_f) = \beta_e \ (t_f) = \hat{\theta}$$

It therefore seems reasonable to approximate β_p and β_e by θ. In Problem 1 of the previous chapter, it is observed that the solutions for all the models yield values of θ that are relatively close. In Problem 2, all solutions yield reasonably close values of θ, excepting the zero induced drag solution without the induced drag correction. For purposes of synthesizing a feedback strategy that is responsive to any move of the opponent, θ must be determined from a dynamic rather than a static model.

The following algorithm is proposed as a closed-loop control law for application with the standard aircraft model. At each integration step

    1. Determine θ using the linearized drag polar model solution.

    2. Determine the instantaneous values of control from

$$C_{Lp} = \tan \ (\theta - \gamma_p)/(2k_p)$$

and

$$C_{Le} = \tan \ (\theta - \gamma_e)/(2k_e)$$

This control law uses the dynamics of the linearized drag polar model and the determination of control offered by the static model. The results of the previous chapter show that the payoff resulting from the static model solutions is closest to the payoff of the standard model in both problems.

For this control law to be "reasonable", the solution to a differential game problem in which the pursuer and evader both employ this control law should have a payoff relatively close to the value of the game. To show this, Problems 1 and 2 defined in the previous chapter are solved. The standard model dynamics are used and the pursuer and evader both employ the closed-loop control law synthesized in this chapter. The solution for Problem 1 is shown in Figure 22 and the solution for Problem 2 is shown in Figure 23. The numerical results are

Problem 1

    Standard Model Extremals - Game Value - 9897 feet

    Standard Model Using
    Synthesized Control Law - Payoff   - 10147 feet

Problem 2

    Standard Model Extremals - Game Value - 5831 feet

    Standard Model Using
    Synthesized Control Law - Payoff   - 6018 feet

It is seen that the application of the synthesized control law does a reasonable job of duplicating the optimal saddle point game solution, implying that the payoff is reasonably close to the value of the game and the trajectories are also close. In order to determine the optimality of this control law, it must be tested against the best
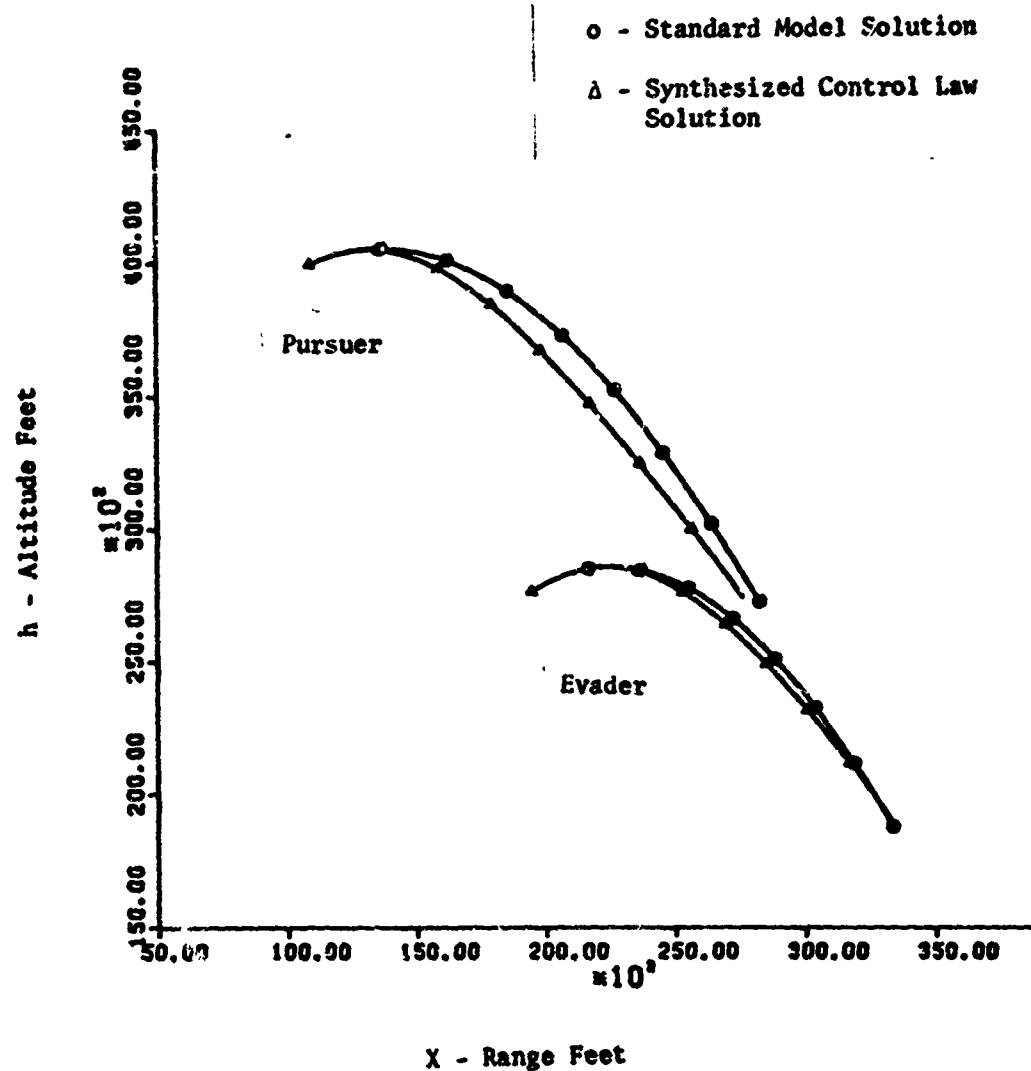
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Figure 22. Standard Model Solution Using the Synthesized
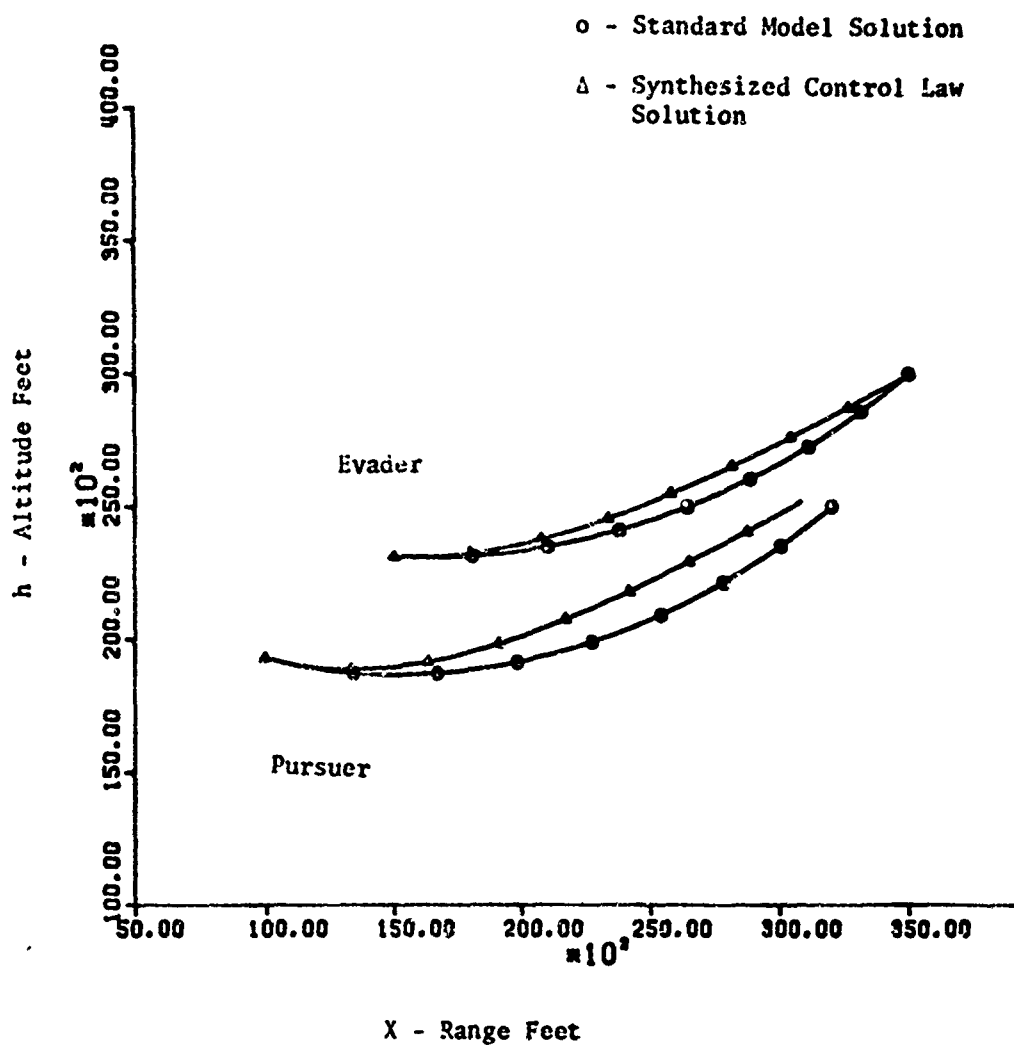Control Law - Problem 1

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Figure 23. Standard Model Solution Using the Synthesized
Control Law - Problem 2

open-loop strategy of the opposing player. Suppose the evader plays optimally against the pursuer who employs this control law and the resulting payoff is equal to the game value plus a distance η. This control law is then said to be η optimal against any strategy for the evader.

Synthesized Control Law Optimality

To determine the optimality of this control law, a computer program was developed that optimizes the evader's trajectory against a pursuer employing this synthesized control law. A parameter optimization problem was formulated by considering the evader's control at each integration time step as an independent parameter.

$$C_{Le}(t_i) = \alpha_i \qquad i = 1 \text{ to } t_f/\Delta t$$

The parameters α_i are then adjusted to maximize the distance between the pursuer and evader at termination $(t = t_f)$. This parameter adjustment was accomplished using the Random Ray Search Algorithm, contained in a program entitled, "Automated Engineering and Scientific Optimization Program (AESOP)", (Ref [7]). This procedure was applied to both problems considered in the preceding chapter. The solutions are shown in Figure 24 and Figure 25. The saddle point solutions are also shown for comparative purposes.

By employing the best open-loop strategy, the evader is able to obtain a payoff of 10541 feet, which represents a six and one-half percent increase over the saddle point game value in Problem 1. In the second problem, the evader is able to obtain a payoff of 6319 feet representing an eight and four-tenths percent increase over the saddle point game value. It must be remembered that these payoffs represent
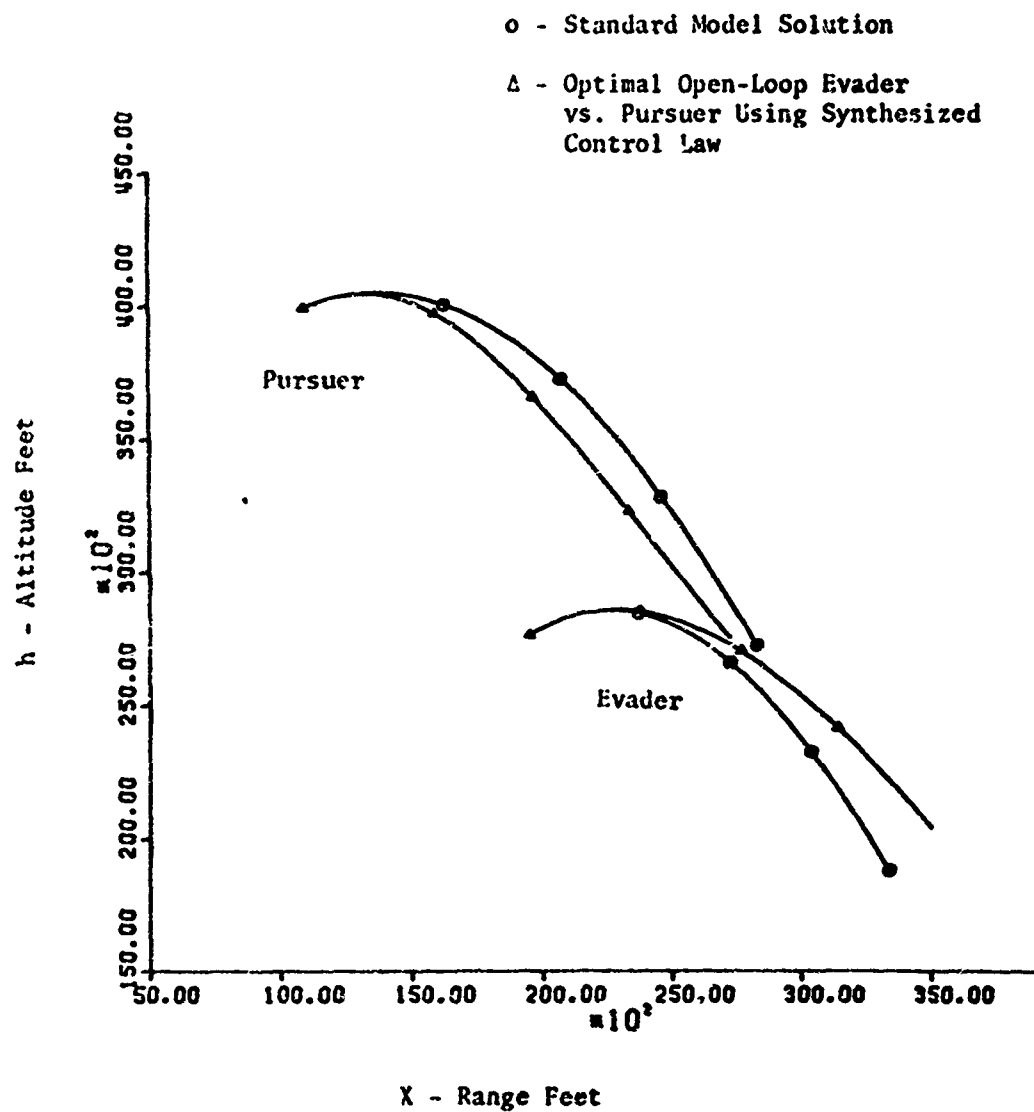
Figure 24. Problem 1 Solution - Evader - Best Open-Loop Strategy
Pursuer - Synthesized Control Law
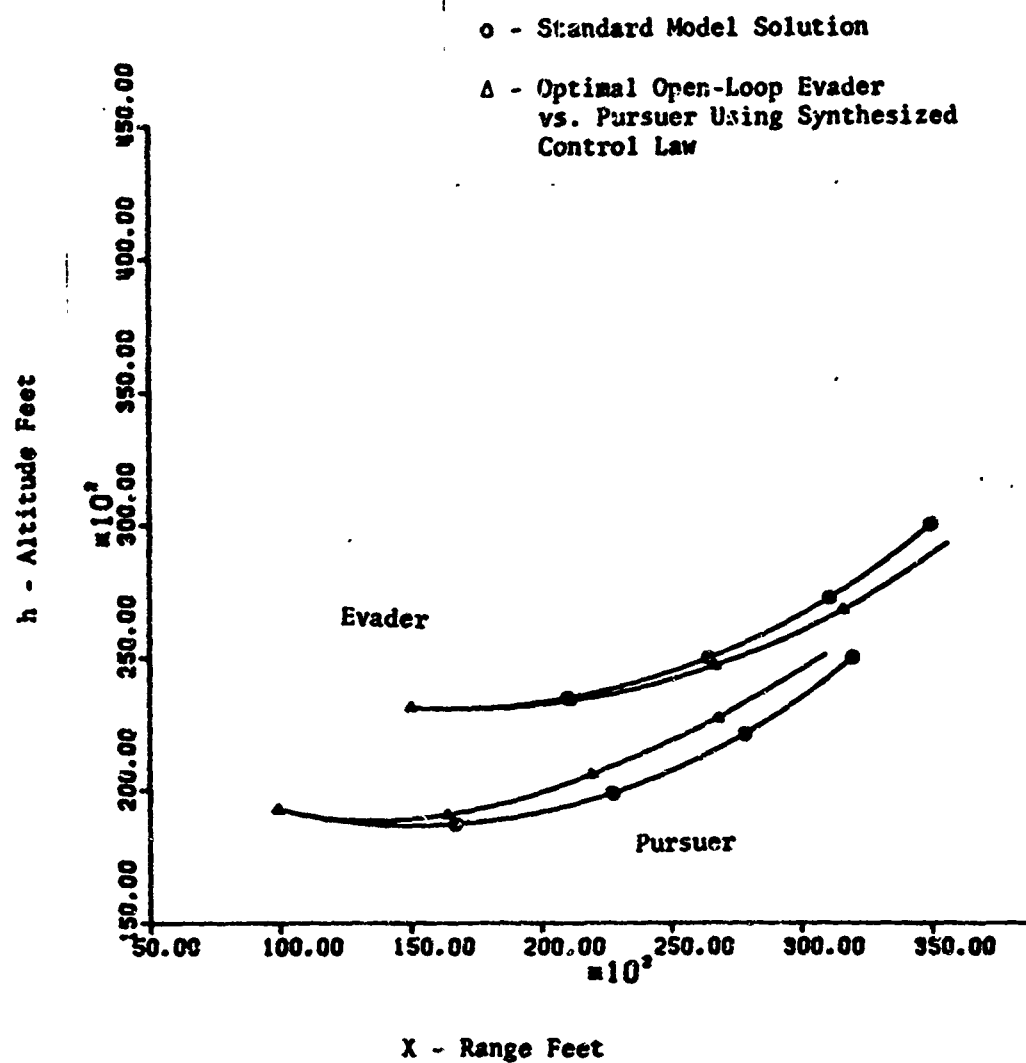
Figure 25.   Problem 2 Solution - Evader - Best Open-Loop Strategy
Pursuer - Synthesized Control Law

the best that the evader can do against the synthesized control law
and that there is no closed-loop control law available that allows the
evader to achieve these payoffs.

X. CONCLUSIONS

Closed-loop solutions to four pursuit-evasion differential games between two aircraft have been presented. For the three simplified aircraft models considered, optimal closed-loop control laws are found. For the standard aircraft model, a "near" or approximate optimal closed-loop control law is synthesized. The optimality of this control law is investigated and found to be quite good.

To be of practical value, the solution to a differential game problem must provide closed-loop control laws. The ability to determine closed-loop control laws through modeling has been demonstrated in this study. It is believed that the same approach can and should be applied to other aspects of the combat differential game with equally fruitful results anticipated.

It is hoped that the approach taken in this investigation will stimulate an interest in the investigation of "practical" differential game problems.

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APPENDIX A: SUFFICIENT CONDITIONS FOR A LOCAL SADDLE POINT

Purpose

The purpose of this appendix is to sketch the derivation of sufficient conditions for a local saddle point solution to the differential game problem considered in this dissertation. The derivation is similar to that in Ref [2], which is analogous to the approach taken for the optimal control problem in Ref [4].

Differential Game Problem

The following differential game problem is considered. Determine a saddle point of

$$J = \phi \ (\underline{x} \ (t_f), \ t_f) + \int_{t_o}^{t_f} L \ (\underline{x}, \ u, \ v, \ t) \ dt \qquad \text{(A-1)}$$

subject to

$$\underline{\dot{x}} = \underline{f} \ (\underline{x}, \ u, \ v, \ t) \qquad \underline{x} \ (t_o) = \underline{x}_o$$

where $\underline{x}$ is an n-vector. The terminal surface is given by

$$\psi \ (\underline{x} \ (t_f), \ t_f) = T - t_f = 0$$

where T is a specified fixed time implying that the game is of finite duration. The functions ϕ, $\underline{f}$, and L are of class C^2 with respect to all arguments. The admissible controls u (t) and v (t) are subject to the constraints

$$u_{min} \leq u \leq u_{max}$$

$$v_{min} \leq v \leq v_{max}$$

and are piecewise continuous on $[t_o, \ t_f]$.

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Necessary Conditions

Adjoining the differential constraints to the payoff J with a Lagrange multiplier $\underline{\lambda}$ (t), Eq (A-1) may be written as

$$J = \phi \; (\underline{x} \; (t_f) \; t_f) + \int_{t_o}^{t_f} \{L \; (\underline{x}, \; u, \; v, \; t) + \underline{\lambda}^T \; (\underline{f} - \underline{\dot{x}})\} \; dt \quad (A-2)$$

The Hamiltonian, H is defined by

$$H \; (t, \; \underline{x}, \; \underline{\lambda}, \; u, \; v) = \underline{\lambda}^T \; \underline{f} + L$$

and it is assumed that for the class of problems being considered the Hamiltonian is separable. This means that H can be separated into two functions, one of which is independent of u and the other independent of v. Integrating the last term on the right side of Eq (A-2) by parts, yields

$$J = \phi \; (\underline{x} \; (t_f), \; t_f) - \underline{\lambda}^T \; (t_f) \; \underline{x} \; (t_f) + \underline{\lambda}^T \; (t_o) \; \underline{x} \; (t_o)$$

$$+ \int_{t}^{t_f} \{H + \underline{\dot{\lambda}}^T \; \underline{x}\} \; dt \quad (A-3)$$

The application of control variations δu and δv results in a variation $\underline{\delta x}$ in the state vector and a change in J. The change in J, to second order, due to these variations may be written as

$$\Delta J = (\phi_{\underline{x}} - \underline{\lambda}^T) \; \underline{\delta x} \; |_{t_f} + (\underline{\lambda}^T \; \underline{\delta x}) \; |_{t_o}$$

$$+ 1/2 \; || \; \underline{\delta x} \; ||^2_{\phi_{\underline{xx}}} \; |_{t_f} + \int_{t_o}^{t_f} [(H_{\underline{x}} + \underline{\dot{\lambda}}^T) \; \underline{\delta x} + H_u \; \delta u$$

$$+ H_v \; \delta v] \; dt + 1/2 \int_{t_o}^{t_f} [\underline{\delta x}^T \; \delta u \; \delta v] \; \hat{H} \begin{bmatrix} \underline{\delta x} \\ \delta u \\ \delta v \end{bmatrix} dt \quad (A-4)$$

where

$$\hat{\underline{H}} = \begin{bmatrix} H_{\underline{xx}} & H_{\underline{x}u} & H_{\underline{x}v} \\ H_{u\underline{x}} & H_{uu} & H_{uv} \\ H_{v\underline{x}} & H_{vu} & H_{vv} \end{bmatrix}$$

and

$|| \underline{\delta x} ||^2_{\phi_{\underline{xx}}}$ denotes the quadratic form $\underline{\delta x}^T \phi_{\underline{xx}} \underline{\delta x}$

$\underline{\delta x}$ is determined from

$$\underline{\dot{\delta x}} = \underline{f}_{\underline{x}} \underline{\delta x} + \underline{f}_u \delta u + \underline{f}_v \delta v \qquad \underline{\delta x}(t_o) = \underline{0} \qquad (A-5)$$

Choosing $\underline{\dot{\lambda}}^T = - H_{\underline{x}}$ and $\underline{\lambda}(t_f) = \phi_{\underline{x}}\Big|_{t_f}$, Eq (A-4) can be written as

$$\Delta J = 1/2 \; || \underline{\delta x} ||^2_{\phi_{\underline{xx}}} \Big|_{t_f} + \int_{t_o}^{t_f} [H_u \delta u + H_v \delta v] \, dt \qquad (A-6)$$

$$+ 1/2 \int_{t_o}^{t_f} [\underline{\delta x}^T \; \delta u \; \delta v] \; \hat{\underline{H}} \begin{bmatrix} \underline{\delta x} \\ \delta u \\ \delta v \end{bmatrix} dt$$

For (u^*, v^*) to provide a saddle point of J, (u^*, v^*) must satisfy

$$J(u^*, v^* + \delta v) \leq J(u^*, v^*) \leq J(u^* + \delta u, v^*) \qquad (A-7)$$

for admissible variations δu and δv.

For Eq (A-7) to hold for any admissible δu and δv, it is necessary from first order considerations of ΔJ that

(i) $H_u = 0 \qquad\qquad H_v = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (A-8)$

(ii) $H_{uu} \geq 0 \qquad\qquad H_{vv} \leq 0$

for $u_{min} < u^* < u_{max}$ and $v_{min} < v^* < v_{max}$. If the minimizing control

u^* and the maximizing control v^* are on the boundaries of their admissible sets it is necessary that

$$H_u \ \delta u \geq 0 \qquad \text{and} \qquad H_v \ \delta v \leq 0 \qquad (A-9)$$

Sufficient Conditions

For non-interior u^* and v^*, $H_u \ \delta u > 0$ and $H_v \ \delta v < 0$ where δu and δv satisfy

$$u_{min} \leq u^* + \delta u \leq u_{max}$$

$$v_{min} \leq v^* + \delta v \leq v_{max} \qquad (A-10)$$

are sufficient to insure that $J(u^*, v^*)$ is at a saddle point since the first order terms in Eq (A-6) dominate for small δu and δv.

For interior u^* and v^* the necessary conditions, Eqs (A-8), cause the first order terms in Eq (A-6) to vanish and the variation in J is given by

$$\delta^2 J = 1/2 \ ||\underline{\delta x}||^2_{\phi_{\underline{xx}}}\Big|_{t_f} + 1/2 \int_{t_o}^{t_f} [\underline{\delta x}^T \ \delta u \ \delta v] \ \hat{\underline{H}} \begin{bmatrix} \delta x \\ \delta u \\ \delta v \end{bmatrix} dt \qquad (A-11)$$

with $\underline{x}$ determined from

$$\dot{\underline{\delta x}} = \underline{f}_{\underline{x}} \ \underline{\delta x} + \underline{f}_u \ \delta u + \underline{f}_v \ \delta v \qquad \underline{\delta x}(t_o) = \underline{0} \qquad (A-12)$$

and all functions are evaluated along the extremal path.

The accessory minimax problem is to determine a saddle point of $\delta^2 J$ given by Eq (A-11) subject to the differential constraint, Eq (A-12). For this problem the minimizing control is δu and the maximizing control is δv and they must satisfy the constraints given by Eqs (A-10).

Assuming that the strengthened Legendre-Clebsch conditions

$$\partial^2 H/\partial u^2 > 0 \qquad\qquad \partial^2 H/\partial v^2 < 0$$

are satisfied and using the results of Ref [4], Eqs (A-11) and (A-12) can be written as

$$\delta^2 J = 1/2 \; || \; \underline{\delta x} \; (t_f) \; ||^2_{\underline{S}} + \int_{t_o}^{t_f} [|| \; \underline{\delta x} \; (t) \; ||^2_{\underline{Q}}$$

$$+ \; \delta u^2 \; R_p - \delta v^2 \; R_e] \; dt \qquad\qquad (A-13)$$

and

$$\underline{\dot{\delta x}} = \underline{F} \; \underline{\delta x} + \underline{G}_p \; \delta u + \underline{G}_e \; \delta v \qquad\qquad \underline{\delta x} \; (t_o) = \underline{0} \qquad (A-14)$$

where

$$\underline{S} \equiv \phi_{\underline{xx}} \Big|_{t_f}$$

$$\underline{Q} \equiv H_{\underline{xx}} - H_{\underline{x}u} H_{uu}^{-1} H_{u\underline{x}} - H_{\underline{x}v} H_{vv}^{-1} H_{v\underline{x}}$$

$$R_p \equiv H_{uu}$$

$$R_e \equiv - H_{vv}$$

$$\underline{F} \equiv \underline{f}_{\underline{x}} - \underline{f}_u H_{uu}^{-1} H_{u\underline{x}} - \underline{f}_v H_{vv}^{-1} H_{v\underline{x}}$$

$$\underline{G}_p \equiv \underline{f}_u$$

$$\underline{G}_e \equiv \underline{f}_v$$

and all functions are evaluated along the extremal path. The solution to this accessory minimax problem, (Ref [2]), yields the optimal strategies

$$\delta u^* = - R_p^{-1} \underline{G}_p^T \underline{\Sigma}(t_f, t) \underline{\delta x}$$

$$\tag{A-15}$$

$$\delta v^* = - R_e^{-1} \underline{G}_e^T \underline{\dot{\Sigma}}(t_f, t) \underline{\delta x}$$

where $\underline{\Sigma}(t_f, t)$ is the symmetric matrix solution of the matrix Riccati equation

$$\underline{\dot{\Sigma}} = - \underline{\Sigma}\,\underline{F} - \underline{F}^T \underline{\delta x} + \underline{\Sigma}\,(\underline{G}_p R_p^{-1} \underline{G}_p^T \tag{A-16}$$

$$- \underline{G}_e R_e^{-1} \underline{G}_e^T)\, \underline{\Sigma} - \underline{Q}$$

where

$$\underline{\Sigma}(t_f, t_f) = \underline{S} \tag{A-17}$$

If Eq (A-16) has a solution defined on $[t_o, t_f)$ satisfying Eq (A-17), then the only solution of Eqs (A-16) satisfying Eq (A-17) is

$$\delta u^*(t) = 0 \qquad \text{and} \qquad \delta v^*(t) = 0$$

Therefore the saddle point solution of the accessory minimax problem is at

$$(\delta u^* = 0,\ \delta v^* = 0) \qquad \text{and} \qquad \delta^2 J(\delta u^*, \delta v^*) = 0$$

since $\underline{\delta x}(t) = \underline{0}$. This implies that

$$\delta^2 J(\delta u \neq 0,\ \delta v^* = 0) > 0$$

and

$$\delta^2 J(\delta u^* = 0,\ \delta v \neq 0) < 0$$

from which it is deduced that

$$J(u^*, v^* + \delta v) \leq J(u^*, v^*) \leq J(u^* + \delta u, v^*)$$

which demonstrates that (u^*, v^*) chosen to satisfy the necessary conditions is a local saddle point for the original differential game problem.

Conclusions

When the controls u^* and v^* are non-interior, the necessary conditions given by Eq (A-9) with the inequalities holding are sufficient for (u^*, v^*) to provide a saddle point of J.

When u^* and v^* are interior controls or non-interior controls with H_u and H_v equal to zero, the following conditions are sufficient for (u^*, v^*) to provide a saddle point for the differential game considered.

(1) The strengthened Legendre-Clebsch conditions

$$H_{uu} > 0 \qquad H_{vv} < 0$$

(2) The existence of a solution to the matrix Riccati equation, Eq (A-16), on the interval $[t_o, t_f)$. In Ref [2], the condition when the solution to the matrix Riccati equation becomes unbounded is related to the existence of a conjugate point on the extremal path.

These sufficiency conditions do not apply to singular arc solutions.

APPENDIX B: DETERMINATION OF ALLOWABLE CONTROL SEQUENCES FOR THE DIFFERENTIAL GAME PROBLEM WITH THE LINEARIZED DRAG POLAR MODEL

Purpose

The purpose of this appendix is to determine the allowable sequences of controls in an optimal solution to the differential game problem using the linearized drag polar model which is presented in Chapter VII.

The state and costate equations for this model are repeated for convenience. The state equations, Eqs (48) and (49) are

$$\dot{x}_p = V_p \cos \gamma_p$$

$$\dot{h}_p = V_p \sin \gamma_p \tag{B-1}$$

$$\dot{V}_p = D_p [C_{Tp} - k_p \mid C_{Lp} \mid]$$

$$\dot{\gamma}_p = D_p C_{Lp}/V_p$$

$$\dot{x}_e = V_e \cos \gamma_e$$

$$\dot{h}_e = V_e \sin \gamma_e \tag{B-2}$$

$$\dot{V}_e = D_e [C_{Te} - k_e \mid C_{Le} \mid]$$

$$\dot{\gamma}_e = D_e C_{Le}/V_e$$

The costate equations, Eqs (50) and (51) with Eqs (62) are

$$\dot{\lambda}_{xp} = 0 \tag{B-3}$$

$$\dot{\lambda}_{hp} = 0$$

$$\dot{\lambda}_{Vp} = \cos(\theta - \gamma_p) + \lambda_{\gamma p} D_p C_{Lp}/V_p^2$$

$$\dot{\lambda}_{\gamma p} = V_p \sin(\theta - \gamma_p)$$

$$\dot{\lambda}_{xe} = 0$$

$$\dot{\lambda}_{he} = 0$$

$$\dot{\lambda}_{Ve} = \cos(\theta - \gamma_e) + \lambda_{\gamma e} D_e C_{Le}/V_e^2 \qquad (B-4)$$

$$\dot{\lambda}_{\gamma e} = V_e \sin(\theta - \gamma_e)$$

In Chapter VII, the controls that minimize the Hamiltonian are determined to be $C_{Lp\ max}$, $C_{Lp\ min}$, $C_{Lp} = 0$, or a singular control C_{Lps}. For the choice $C_{Lp\ max}$ to be optimal

$$M_p^+ = D_p [-\lambda_{Vp} k_p + \lambda_{\gamma p}/V_p] < 0 \qquad (B-5)$$

and for $C_{Lp\ min}$ to be optimal

$$M_p^- = D_p [\lambda_{Vp} k_p + \lambda_{\gamma p}/V_p] > 0 \qquad (B-6)$$

For $C_{Lp} = 0$ to be optimal, M_p^+ must be positive or M_p^- must be negative. The singular control C_{Lps} occurs if M_p^+ or M_p^- is zero over a finite time interval.

Allowable Control Sequences

In determining allowable sequences of controls, the physical aspect of the problem must be considered. It is pointed out in Chapter VII, that there is a direction θ associated with the solution to this problem and that this direction is the line of sight direction

between the two aircraft at $t = t_f$, when they both play optimally. Since this θ direction exists for optimal play, it is physically logical to assume that any control that causes either aircraft to turn away from this direction is non-optimal. Conversely, any non-zero control to be optimal must rotate γ_p and γ_e toward the θ direction.

Consider initial conditions such that the minimizing control for the pursuer is initially $C_{Lp\ min}$. For these initial conditions

$$M_p^- = D_p [\lambda_{Vp} k_p + \lambda_{\gamma p}/V_p] > 0$$

from Eq (B-6). For $C_{Lp\ min}$ to be an optimal control, the physical aspects of the problem require $(\theta - \gamma_p)$ to be negative. The allowable sequences of controls from these initial conditions can be determined by investigating the behavior of the switching function M_p^-. Figure 26, shows a switching function trajectory corresponding to the control sequence $\{C_{Lp\ min}, C_{Lp} = 0, C_{Lp\ min}\}$. $M_p^-(t_f)$ equals zero since $\lambda_{Vp}(t_f)$ and $\lambda_{\gamma p}(t_f)$ are zero from transversality conditions, Eqs (52). To show that this sequence is not an optimal sequence, the derivative of $\dot{M}_p^-$ is

$$\dot{M}_p^- = D_p [\dot{\lambda}_{Vp} k_p + \dot{\lambda}_{\gamma p}/V_p - \lambda_{\gamma p} \dot{V}_p/V_p^2] \tag{B-7}$$

Substituting for $\dot{\lambda}_{Vp}$, $\dot{\lambda}_{\gamma p}$, and $\dot{V}_p$ using Eqs (B-1) and (B-3), Eq (B-7) becomes

$$\dot{M}_p^- = D_p [k_p \cos(\theta - \gamma_p) + \sin(\theta - \gamma_p) - \lambda_{\gamma p} D_p C_{Tp}/V_p^2] \tag{B-8}$$

It is seen in Figure 26, that to end on a $C_{Lp\ min}$ arc, $\dot{M}_p^-(t_f)$ must be negative. This implies from Eq (B-8) that
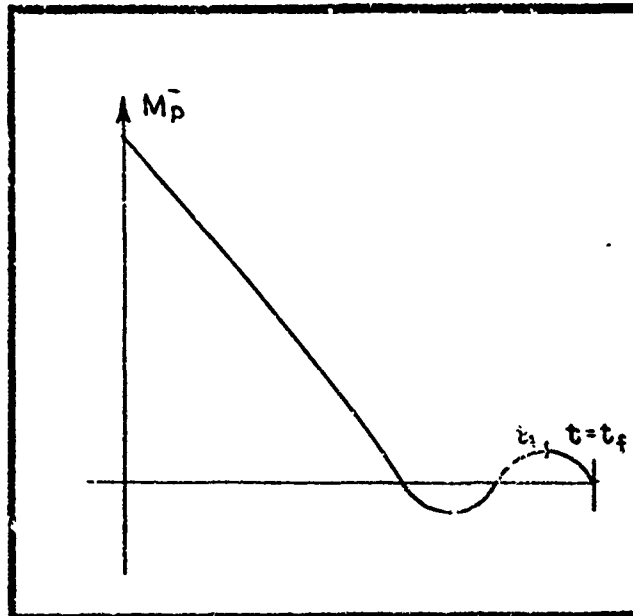
Figure 26. Switching Function Trajectory for the Control Sequence
$$\{C_{Lp\ min}, \ C_{Lp} = 0, \ C_{Lp\ min}\}$$

$$\tan(\theta - \gamma_p)\ \Big|_{t_f} < - k_p \qquad (B-9)$$

since $\lambda_{\gamma p}(t_f) = 0$ from transversality conditions. For the sequence
$\{C_{Lp\ min}, \ C_{Lp} = 0, \ C_{Lp\ min}\}$ to exist, $\dot{M}_p^-$ must be zero at $t_1 < t_f$.
From Eq (B-8) this occurs when

$$k_p \cos(\theta - \gamma_p) + \sin(\theta - \gamma_p) = \lambda_{\gamma p}\ D_p\ C_{Tp}/V_p^2 \qquad (B-10)$$

It has been stated that the physical aspects of the problem require
$(\theta - \gamma_p)$ to be negative for the control $C_{Lp\ min}$ to be optimal. Eq
(B-9) confirms this requirement. For negative $(\theta - \gamma_p)$, $\dot{\lambda}_{\gamma p}$ is
negative and $\lambda_{\gamma p}$ must be positive in order to satisfy $\lambda_{\gamma p}(t_f) = 0$.
This implies that the right hand side of Eq (B-10) is positive and
that

$$\tan{(\theta - \gamma_p)} \Big|_{t_1} > - k_p \tag{B-11}$$

Eqs (B-10) and (B-11) imply that γ_p has increased with time which is not possible with the control $C_{Lp\ min}$. It is concluded that the control sequence $\{C_{Lp\ min}, \ C_{Lp} = 0, \ C_{Lp\ min}\}$ is not an optimal sequence. A more basic conclusion is that $\dot{M}_p^-$ cannot be zero for $t < t_f$ if the solution is to end on a $C_{Lp\ min}$ arc. This result is used in the following analysis.

Singular Controls

The question whether or not singular controls exist for this problem is now considered. Initial conditions are again chosen such that the minimizing control for the pursuer is initially $C_{Lp\ min}$. Figure 27 shows three switching function trajectories containing singular controls.
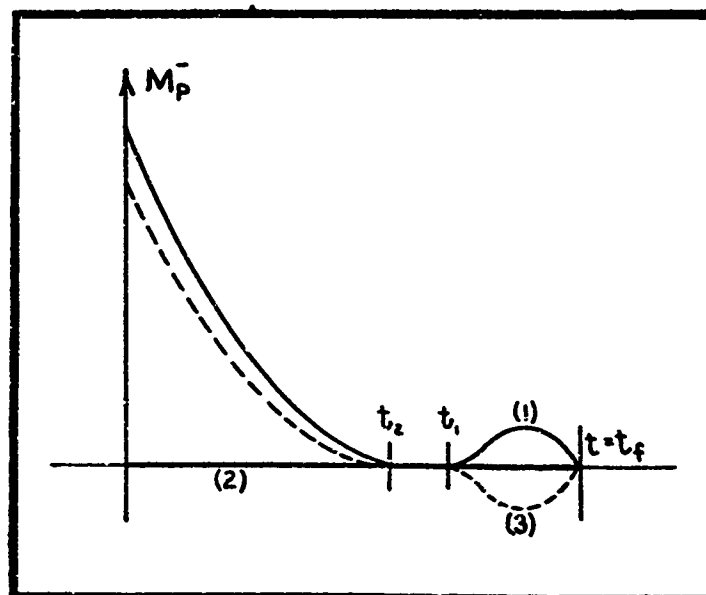

Figure 27. Switching Function Trajectories Containing Singular Arcs

Trajectory 1, in Figure 27, corresponds to the control sequence $\{C_{Lp\ min}, C_{Lps}, C_{Lp\ min}\}$. This sequence can be eliminated as a possible optimal sequence by the results previously obtained which conclude that $\dot{M}_p^-$ cannot be zero for $t < t_f$ if the solution is to end on a $C_{Lp\ min}$ arc.

Trajectory 2, in Figure 27, corresponds to the control sequence $\{C_{Lps}\}$ and trajectory 3 corresponds to the sequence $\{C_{Lp\ min}, C_{Lps}, C_{Lp} = 0\}$.

The sequence $\{C_{Lps}\}$ implies that M_p^- and $\dot{M}_p^-$ are zero along the singular arc. To show that this is not an optimal sequence, setting $\dot{M}_p^-(t_f)$ equal to zero implies

$$\tan(\theta - \gamma_p)\Big|_{t_f} = -k_p \qquad (B\text{-}12)$$

Eq (B-12) implies that $(\theta - \gamma_p)$ is negative and from Eq (B-3) $\dot{\lambda}_{\gamma p}$ is negative. Therefore, for $t < t_f$, $\lambda_{\gamma p}$ must be positive to satisfy $\lambda_{\gamma p}(t_f) = 0$. $\dot{M}_p^-$ must be zero for all t. For $t < t_f$ Eq (B-8) implies

$$k_p \cos(\theta - \gamma_p) + \sin(\theta - \gamma_p) = \lambda_{\gamma p} D_p C_{Tp}/V_p^2 > 0 \qquad (B\text{-}13)$$

Eq (B-13) yields

$$\tan(\theta - \gamma_p)\Big|_{t < t_f} > -k_p \qquad (B\text{-}14)$$

Eqs (B-12) and (B-14) imply that the singular control has rotated the flight path direction of the pursuer away from the final line of sight direction θ. It is concluded that an optimal solution to this problem cannot terminate on a singular arc.

The remaining control sequence to consider is $\{C_{Lp\ min}, C_{Lps}, C_{Lp} = 0\}$ which corresponds to trajectory 3 in Figure 27. Let $t = t_1$, denote the time of switching from the singular to the $C_{Lp} = 0$ arc.

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For $t_2 < t < t_1$, $\dot{M}_p^-$ must equal zero. This implies from Eq (B-8) that

$$\tan (\theta - \gamma_p) \Big|_{t<t_1} > -k_p \tag{B-15}$$

since $\lambda_{\gamma p} D_p \dot{C}_{Tp}/V_p^2$ is positive for reasons already stated. For

$t > t_1$, $\dot{M}_p^-$ must be negative from Figure 27. This implies, using Eq

(B-8), that

$$\tan (\theta - \gamma_p) \Big|_{t>t_1} < -k_p \tag{B-16}$$

Eqs (B-15) and (B-16) imply that the singular control has rotated the

flight path direction away from the θ direction. This control sequence

is therefore not optimal.

Similar arguments can be made using M_p^+ and $\dot{M}_p^+$ to show that the

sequences

$$\{C_{Lp\ max}, C_{Lp} = 0, C_{Lp\ max}\}, \{C_{Lp\ max}, C_{Lps}, C_{Lp\ max}\},$$

and

$$\{C_{Lp\ max}, C_{Lps}, C_{Lp} = 0\}$$

are not optimal control sequences. Control sequences involving

switching between maximum and minimum controls are not optimal because

they imply a rotation away from the terminal line of sight direction

θ.

Conclusions

The conclusion is that for the pursuer the following control

sequences are candidates for an optimal solution:

$$\{C_{Lp\ max}\ or\ C_{Lp\ min}, C_{Lp} = 0\}, \{C_{Lp\ max}\ or\ C_{Lp\ min}\}, \{C_{Lp} = 0\}$$

An analysis similar to this for the evader leads to the conclusion that an optimal solution consists of one of the following control sequences:

$$\{C_{Le \ max} \ or \ C_{Le \ min}, \ C_{Le} = 0\}, \ \{C_{Le \ max} \ or \ C_{Le \ min}\}, \ \{C_{Le} = 0\}$$

VITA

William L. Othling Jr. was born on 17 April 1934 in Cincinnati, Ohio. He graduated from high school in Cincinnati in 1952 and attended the University of Cincinnati from which he received the degree of Mechanical Engineering and a commission in the USAF in 1957. He served with the Alaskan Air Command and the Strategic Air Command until he entered Oklahoma State University where he received the Master of Science degree in Mechanical Engineering in 1962. He was assigned to the Oklahoma Air Materiel Area, Oklahoma City, Oklahoma where he served as a Supervisory Mechanical Engineer until his entry into the Air Force Institute of Technology in July, 1965. He is presently an Aerospace Engineer in the High Speed Aero-Performance Branch, Flight Mechanics Division, Air Force Flight Dynamics Laboratory, Wright Patterson Air Force Base, Ohio.

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13. ABSTRACT

The pursuit-evasion aspect of the two aircraft combat problem is introduced as a fixed time, zero sum, perfect information differential game. A realistic aircraft model is presented for which a solution of this combat problem is desired. Because of the non-linear dynamics associated with this model, an optimal closed-loop solution cannot be obtained. Three additional simplified aircraft models are introduced as approximations to the realistic model. Optimal solutions and closed-loop control laws are obtained for each of these models. Analysis of these solutions and control laws enables the formulation of an approximate closed-loop control law for use with the original realistic model.

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